Book of abstracts

Regularity theory for free boundary and geometric variational problems III

Levico Terme (Italy), 19-23 June 2023

Boundary Harnack principles and degenerate equations on singular sets

Susanna Terracini

The ratio v/u of two solutions to a second order elliptic equation in divergence form solves a degenerate elliptic equation if u and v share the zero set; that is, $Z(u) \subset Z(v)$. The coefficients of the degenerate equation vanish on the nodal set as u. Developing a Schauder theory for such equations, we prove $C^{k,\alpha}$-regularity of the ratio from one side of the regular part of the nodal set in the spirit of the higher order boundary Harnack principle established by De Silva and Savin in [4]. Then, by a gluing lemma, the estimates extend across the regular part of the nodal set. Eventually, using conformal mapping in dimension $n = 2$, we provide local gradient estimates for the ratio which hold also across the singular part of the nodal set and depends on the highest value attained by the Almgren frequency function. This talk is based on joint works with Giorgio Tortone and Stefano Vita.

[1] S. Terracini, G. Tortone, S. Vita. *Higher order boundary Harnack prin- ciple on nodal domains via degenerate equations.* Preprint, 2022.

[2] Y. Sire, S. Terracini, S. Vita. *Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions.* Comm. PDE 46-2 (2021), 310-361.

[3] Y. Sire, S. Terracini, S. Vita. *Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions.* Math. Engineering, 3-1 (2021), 1-50.

[4] D. De Silva, O. Savin. *A note on higher regularity boundary Harnack inequality.* DCDS-A, 35(12), (2015) 6155–6163.

Compact contact sets in the thin obstacle problem

Hui Yu

With the recent classification of contact sets of global solutions to the obstacle problem, it is natural to study the analogue question for the thin obstacle problem. In this talk, we describe the classification of compact contact sets for solutions with at most quadratic growth. This is based on a joint work with Simon Eberle.

On the closability of differential operators

Andrea Marchese

One way of defining Sobolev functions in $W^{1,p}$ and their weak gradient in the Euclidean space is to consider the closure in $L^p \times L^p$ of the graph of the gradient operator defined on the space of functions of class C^1 and to prove that this is also a graph. Extending such construction, one says that an operator is closable if the closure of its graph is a graph.

For any Radon measure μ on the Euclidean space, I will show how to characterize the directions for which the corresponding derivatives are closable from the space of Lipschitz functions to $L^p(\mu)$. I will also discuss some partial results on the closability of directional derivatives from $L^p(\mu)$ to $L^q(\mu)$ and of some multilinear operators and how to interpret these results in terms of the structure of Ambrosio-Kirchheim metric currents.

On the regularity of the optimal shapes for a class of integral functionals

Giorgio Tortone

The talk deals with the regularity of a free boundary problem arising in the optimization of a class of integral shape functionals. Three variables are involved: two state function u, v and a shape Ω, with u and v satisfying an overdetermined boundary value problem involving the product of their normal derivatives.

The key points of the analysis are a blow-up analysis, involving three blow-ups before finding a homogeneous limit function, and the study of the dimension of the singular set, which requires a new theory for stable solutions of the Bernoulli problem.

These results have been obtained in collaboration with G. Buttazzo, F. Maiale, D. Mazzoleni, and B. Velichkov.

Almost minimizers of a lower-dimensional free boundary problem

Mariana Smit Vega Garcia

In this talk, we will consider almost minimizers of a two-phase free boundary problem given by an energy functional connected to the fractional Laplacian. We prove regularity of almost minimizers, and show that the two free boundaries cannot touch. Finally, we will discuss the regularity of the free boundary. This is joint work with Mark Allen

Area variations under lagrangian and legendrian constraints

Tristan Rivière

In any 5 dimensional closed Sasakian manifold we prove that any minmax operation on the area among Legendrian surfaces is achieved by a continuous conformal Legendrian map from a closed riemann surface S into N^5 equipped with an integer multiplicity bounded in L^∞. Moreover this map, equipped with this multiplicity, satisfies a weak version of the Hamiltonian Minimal Equation. We conjecture that any solution to this equation is a smooth branched Legendrian immersion away from isolated Schoen-Wolfson conical singularities with non zero Maslov class.

Hypersurfaces with mean curvature prescribed by an ambient function

Costante Bellettini

Let N be a compact Riemannian manifold of dimension 3 or higher, and g a Lipschitz non-negative (or non-positive) function on N. In joint works with Neshan Wickramasekera, after developing a suitable compactness and regularity theory, we prove that there exists a two-sided (well-defined global unit normal) closed hypersurface M whose mean curvature attains the values prescribed by g. Except possibly for a small singular set (of codimension 7 or higher), the hypersurface M is C^2 immersed; more precisely, the immersion is a quasi-embedding, namely the only non-embedded points are caused by tangential self-intersections (around such a non-embedded point, the local structure is given by two disks, lying on one side of each other, and intersecting tangentially, as in the case of two spherical caps touching at a point). The proof employs an Allen–Cahn approximation scheme, classical minmax, and gradient flow arguments. A key issue that appears in this construction, as well as in related compactness questions for PMC hypersurfaces, is the possible formation of "hidden boundaries": we will explore recent progress in these directions.

Existence and regularity of Brakke flows: new advances and open questions

Salvatore Stuvard

The Brakke flow is a measure-theoretic generalisation of the Mean Curvature Flow which allows to describe the evolution by mean curvature of surfaces admitting singularities and topology changes. A typical example is that of "grain boundaries": the interfaces separating different grains or phases in a material (e.g. a polycrystal) and evolving in time subject to a potential energy of surface tension type.

In this talk, I will discuss some recent advances in the theory concerning existence, qualitative properties, and regularity of Brakke flows, and, if time permits, some applications to Plateau?s problem and the regularity theory for minimal surfaces.

Based on joint works with Yoshihiro Tonegawa (Tokyo Inst. of Tech.).

Higher codimension area-minimizers: rectifiability of singularities and $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones

Anna Skorobogatova

The problem of determining the size and structure of the interior singular set of area-minimizing surfaces has been studied thoroughly in a number of different frameworks, with many ground-breaking contributions. In the framework of integral currents, when the codimension of the surface is higher than 1, the presence of singular points with flat tangent cones creates an obstruction to easily understanding the interior singularities. Little progress has been made in full generality since the celebrated $(m-2)$-Hausdorff dimension bound on the singular set due to Almgren, which was since revisited and simplified by De Lellis and Spadaro.

In this talk I will discuss recent joint works with Camillo De Lellis and Paul Minter, where we establish $(m-2)$-rectifiability of the interior singular set of an m-dimensional integral current and show that the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. interior point.

Nondegenerate minimal submanifolds as energy concentration sets

Alessandro Pigati

Various energies of physical significance have been shown to effectively approximate the area functional, in codimension one and two. These energies are defined on the set of functions on a given ambient manifold, and for critical maps the energy density tends to concentrate towards a (possibly singular) minimal submanifold, as a suitable scaling parameter goes to zero. In this talk we solve the converse problem: we show that any nondegenerate minimal submanifold of the corresponding codimension does arise in this way. The strategy is entirely variational and generalizes a recent work for geodesics (by Colinet, Jerrard, and Sternberg), by extending two key geometric measure theory results to arbitrary dimension. (Joint work with Guido De Philippis)

Obstacle problems for fractional powers of the Laplacian

Donatella Danielli

In this talk we will discuss a sampler of obstacle-type problems associated with the fractional Laplacian $(?\Delta)^s$, for $1 < s < 2$.

Our goals are to establish regularity properties of the solution and to describe the structure of the free boundary. To this end, we combine classical techniques from potential theory and the calculus of variations with more modern methods, such as the localization of the operator and monotonicity formulas. This is joint work with A. Haj Ali (Arizona State University) and A. Petrosyan (Purdue University).

On global graphical solutions to free boundary problems

Max Engelstein

The Bernstein problem for minimal surfaces asks whether a globally defined minimal hypersurface given by the graph of a function in dimension n must be a hyperplane. This was resolved by the combined work of De Giorgi, Simons and then De Giorgi-Bombieri-Giusti; showing that the answer is yes when $n \leq 8$ and no when $n \geq 9$.

In this talk we will discuss recent progress towards this question for one-phase free boundary problems of Bernoulli type. This is joint with Xavier Fernandez-Real (EPFL) and Hui Yu (NUS).

Stationary and rotating spirals in segregated reaction-diffusion systems

Gianmaria Verzini

In the first part of this talk, I will describe the structure of the nodal set of segregation profiles arising in the singular limit of planar, stationary reaction-diffusion systems with strongly competitive interactions of Lotka-Volterra type, when the matrix of the inter-specific competition coefficients is asymmetric and the competition parameter tends to infinity. Unlike the symmetric case, when it is known that the nodal set consists in a locally finite collection of curves meeting with equal angles at a locally finite number of singular points, the asymmetric case shows the emergence of spiraling nodal curves, still meeting at locally isolated points with finite vanishing order.

In the second part I will investigate the existence of solutions to the corresponding evolutive system, having the same nodal structure and rotating with constant angular speed. This are joint works with Susanna Terracini (Torino), Alessandro Zilio (Paris Cité) and the latter also with Ariel Salort (Buenos Aires).

Interior regularity for two-dimensional stationary Q-valued maps

Jonas Hirsch

We consider 2-dimensional Q-valued maps that are stationary with respect to outer and inner variations of the Dirichlet energy. In this talk I would like to present some ideas that can be used to show that these maps are actually Hölder continuous and that the dimension of their singular set is at most one.

I would like to highlight how one can adapt ideas of M. Grüter to ?reverse? the Douglas and Rado approach. We will use properties of minimal surfaces to show continuity of stationary ?harmonic? functions. This idea helps us to localise in the target space AQ and overcome the problem of the absence of an ?honest? PDE.

If time permits I would like to discuss problems that arise transferring the idea to the non-linear setting of multi-graphs, currents or even varifolds that are stationary for the area.

Generic regularity of minimizing hypersurfaces in dimensions 9 and 10

Christos Mantoulidis

In joint work with Otis Chodosh and Felix Schulze we showed that the problem of finding a least-area compact hypersurface with prescribed boundary or homology class has a smooth solution for generic data in dimensions 9 and 10. In this talk I will explain the main steps of the proof.

Hyperbolic groups and spherical minimal varieties

Antoine Song

I will describe some new examples of non-compact minimal varieties inside the Hilbert sphere minus a discrete subset. Our construction is related to a topological invariant introduced by Besson-Courtois-Gallot called the spherical volume.

Local minimizers of the anisotropic isoperimetric problem on closed manifolds

Robin Neumayer

We show that local minimizers for the anisotropic isoperimetric problem in the small volume regime on closed Riemannian manifolds are small smooth perturbations of tangent Wul ff shapes, quantitatively in terms of the volume. This talk is based on joint work with Antonio De Rosa.

Non-classical solutions to the p-Laplace equation

Riccardo Tione

In this talk we will consider the p-Laplace equation, $div(|Du|^{p-2}Du) = 0$. In particular, we will focus on very weak solutions, i.e. solutions u to the p-Laplace equation with $u \in W^{1,q}$, $\max\{1, p-1\} < q < p$. In 1994, T. Iwaniec and C. Sbordone showed that if q is sufficiently close to p, then very weak solutions belong to $W^{1,p}$, and thus are classical solutions. They conjectured the same to happen for any $\max\{1, p-1\} < q$. In this talk, I will present a positive result which shows that Iwaniec and Sbordone's conjecture is true if the gradient of u belongs to suitable cones, and next I will sketch the construction of a counterexample for this conjecture if this additional condition is not fulfilled. This is based on a joint work with Maria Colombo.

Some theorems on diffused interefaces

Francesco Maggi

In this talk we present several new results concerning the Allen?Cahn energy functional and the modeling of diffused interfaces: (1) a diffused interface Euclidean isoperimetric theorem with a sharp quantitative stability inequality independent of the transition length scale; (2) a diffused interface Alexandrov's type theorem characterizing global minimizers of the diffused Euclidean isoperimetric problems as the only volume-constrained critical points; (3) a description of the long time behavior of the volume preserving diffuse mean curvature flow, with exponential decay rates toward a single diffused ball; (4) the approximation of Plateau-type singularities by solutions of the Allen-Cahn equation with homotopic spanning constraint. Results (1) and (2) have appeared in published work by Maggi and Restrepo on Analysis & PDE. Result (3) and (4) are contained in forthcoming papers by Bonforte, Maggi and Restrepo, and by Maggi, Novack, and Restrepo, respectively.

Counter-example in boundary unique continuations

Zihui Zhao

Unique continuation property is a fundamental property for harmonic functions, as well as a large class of elliptic and parabolic PDEs. It says that if a harmonic function vanishes at a point to infinite order, it must vanish everywhere. In the same spirit, we are interested in quantitative unique continuation problems, where we use the growth rate of a harmonic function to deduce some global estimates, such as estimating the size of its singular set. In this talk, I will talk about some boundary unique continuation results, and show that these results are sharp by giving explicit examples using harmonic measures. This is joint work with C. Kenig.

Geometric structure for nodal sets of solutions to degenerate and singular equations

Yannick Sire

I will report on some recent results on the stratification properties of the nodal sets of solutions to a rather general class of degenerate/singular equations. Such equations are ubiquitous in anomalous diffusions, water wave problems with free boundary, or the geometry of Poincare-Einstein manifolds just to name a few. I will explain how to derive several measure-theoretic properties of the nodal sets and present some open problems.