Book of abstracts

Regularity theory for free boundary and geometric variational problems

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Free boundaries in segregation problems

Susanna Terracini

We first consider classes of variational problems for densities that repel each other at distance. Examples are given by the minimizers of Dirichlet functional or the Rayleigh quotient

$$D(\mathbf{u}) = \sum_{i=1}^{k} \int_{\Omega} |\nabla u_i|^2 \quad \text{or} \quad R(\mathbf{u}) = \sum_{i=1}^{k} \frac{\int_{\Omega} |\nabla u_i|^2}{\int_{\Omega} u_i^2}$$

over the class of $H^1(\Omega, \mathbb{R}^k)$ functions attaining some boundary conditions on $\partial\Omega$, and subjected to the constraint

$$\text{dist}(\{u_i > 0\}, \{u_j > 0\}) \geq 1 \qquad \forall i \neq j.$$

As second class of problems, we consider the problem of spiralling solutions to segregated limiting profiles of competition-diffusion systems

For these problems, we investigate the optimal regularity of the solutions, prove a free-boundary extremality condition, derive some preliminary results characterising the emerging free boundary and finally we will discuss the open problems.

Area minimizing currents mod p: general regularity theory

Andrea Marchese

I will discuss a partial regularity theorem for area minimizing currents mod p, valid for every p, in any dimension and codimension. Our main result establishes that the interior singular set of m-dimensional area minimizing currents mod p has at most Hausdorff dimension $m-1$. Moreover it is rectifiable and of locally finite Hausdorff measure for odd values of p. Based on joint works with C. De Lellis, J. Hirsch, and S. Stuvard.

The Bernstein problem for equations of minimal surface type

Connor Mooney

The Bernstein problem asks whether entire minimal graphs in dimension $N+1$ are necessarily hyperplanes. This problem was solved in combined works of Bernstein, Fleming, De Giorgi, Almgren, and Simons ("yes" if $N < 8$), and Bombieri-De Giorgi-Giusti ("no" otherwise). We will discuss the analogue of this problem for graphical minimizers of anisotropic energies. In particular, we will discuss new examples of nonlinear entire graphical minimizers in the case $N = 6$, and recent joint work with Y. Yang towards constructing such examples in the lowest-possible-dimensional case $N = 4$.

Interface regularity and 1D symmetry for semilinear one-phase problems

Alessandro Audrito

We study critical points of a 1-parameter family of functionals arising in combustion models which converge, for infinitesimal values of the parameter, to the one-phase free boundary problem. We prove a $C^{1,\alpha}$ estimates for the "interfaces" (level sets separating the *burnt* and *unburnt* regions). As a byproduct, we obtain the one-dimensional symmetry of minimisers in the whole $\mathbb{R}^N$, for $N \leq 4$.

Our results are to Bernoulli's free boundary problem what Savin's results for the Allen-Cahn equation are to minimal surfaces. This is a joint work with J. Serra (ETH).

The capillarity one-phase problem

Giorgio Tortone

In this talk we introduce a class of one-phase free boundary problem with a non-homogeneous Neumann condition on a fixed boundary. The plan is to understand how the "permeable" attitude of the fixed boundary affects the regularity and geometry of the arising free boundary. More precisely, we discuss the stratification of the restriction of the free boundary on the fixed wall and we present an epsilon-regularity theorem as well as the non-existence of singular stable cones in lower dimensions.

This talk is based on a joint work with B. Velichkov.

Optimal transport, weak Laplacian bounds and minimal boundaries in non-smooth spaces with lower Ricci curvature bounds

Andrea Mondino

The goal of the seminar is to report on recent joint work with Daniele Semola, motivated by a question of Gromov to establish a "synthetic regularity theory" for minimal surfaces in non-smooth ambient spaces. In the setting of non-smooth spaces with lower Ricci Curvature bounds:

- We establish a new principle relating lower Ricci Curvature bounds to the preservation of Laplacian bounds under the evolution via the Hopf-Lax semigroup;
- We develop an intrinsic viscosity theory of Laplacian bounds and prove equivalence with other weak notions of Laplacian bounds;
- We prove sharp Laplacian bounds on the distance function from a set (locally) minimizing the perimeter: this corresponds to vanishing mean curvature in the smooth setting;
- We study the regularity of boundaries of sets (locally) minimizing the perimeter, obtaining sharp bounds on the Hausdorff co-dimension of the singular set plus content estimates and topological regularity of the regular set.

Optimal transport plays the role of underlying technical tool for addressing various points.

Generic regularity in obstacle problems

Alessio Figalli

The classical obstacle problem consists of finding the equilibrium position of an elastic membrane whose boundary is held fixed and which is constrained to lie above a given obstacle. By classical results of Caffarelli, the free boundary is C^∞ outside a set of singular points. Explicit examples show that the singular set could be in general $(n-1)$-dimensional –that is, as large as the regular set. In a recent paper with Ros-Oton and Serra we show that, generically, the singular set has zero $\mathcal{H}^{n-4}$ measure (in particular, it has codimension 3 inside the free boundary), solving a conjecture of Schaeffer in dimension $n \leq 4$. The aim of this talk is to give an overview of these results.

Charged liquid drop

Domenico Angelo La Manna

We study the motion of an incompressible electrically charged droplet in vacuum. The internal stress causes the droplet to move and our goal is to study the equations of motion. The study of such a problem brings us to investigate the properties of a solution to Euler equation with free boundary.

Strong unique continuation and local asymptotics at the boundary for fractional elliptic equations

Stefano Vita

We study local asymptotics of solutions to fractional elliptic equations at boundary points, under some outer homogeneous Dirichlet boundary condition. Our analysis is based on a blow-up procedure which involves some Almgren type monotonicity formulæ and provides a classification of all possible homogeneity degrees of limiting entire profiles. As a consequence, we establish a strong unique continuation principle from boundary points. This is a joint work with A. De Luca and V. Felli

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A perturbative estimates approach to the free boundary regularity in the one-phase Stefan problem

Nicolò Forcillo

In Stefan type problems, free boundaries may not regularize instantaneously. There exist in particular examples in which Lipschitz free boundaries preserve corners. Nevertheless, in the two-phase Stefan problem, Athanasopoulos, Caffar?elli, and Salsa showed ?that Lipschitz free boundaries in space-time become smooth under a nondegeneracy condition, as well as sufficiently “flat" ones. Their techniques are based on the original work of Caffarelli in the elliptic case. In the talk, we will present a more recent approach to study the regularity of flat free boundaries for the one-phase Stefan problem. It relies on perturbation arguments leading to a linearization of the problem, in the spirit of the elliptic counterpart already developed by Daniela De Silva. This talk is based on a joint work with Daniela De Silva and Ovidiu Savin.

Stable cones in the thin one-phase free boundary problem

Xavier Ros-Oton

We study homogeneous stable solutions to the thin (or fractional) one-phase free boundary problem. The problem of classifying stable (or minimal) homogeneous solutions in dimensions $n \geq 3$ is completely open. In this context, axially symmetric solutions are expected to play the same role as Simons? cone in the classical theory of minimal surfaces, but even in this simpler case the problem is open. The goal of this talk is to present some new results in this direction.

On the one hand we find, for the first time, the stability condition for the thin one-phase problem. Quite surprisingly, this requires the use of “large solutions" for the fractional Laplacian, which blow up on the free boundary.

On the other hand, using our new stability condition, we show that any axially symmetric homogeneous stable solution in dimensions $n < 6$ is one-dimensional, independently of the parameter $s \in (0, 1)$.

Area minimizing hypersurfaces mod(p): A geometric free boundary problem

Jonas Hirsch

In this talk I would like to give an idea of our resent result on the structure of area minimizing hypersurfaces mod(p).

Motivation: If one considers real soap films one notice that from time to time one can find configurations where different soap films join on a common piece. One possibility to allow this kind of phenomenon is to consider flat chains with coefficients in $\mathbb{Z}_p$. For instance for $p = 2$ one can deal with unoriented surfaces, for $p = 3$ one allows triple junctions. Using known results it can be shown that for $p = 3$ this common piece is itself nicely regular. It was our aim to investigate the situation for higher p.

We consider area minimizing m-dimensional currents mod(p) in complete C^2 Riemannian manifolds Σ of dimension $m+1$. For odd moduli we prove that, away from a closed rectifiable set of codimension 2, the current in question is, locally, the union of finitely many smooth minimal hypersurfaces coming together at a common $C^{1,\alpha}$ boundary of dimension $m-1$, and the result is optimal. For even p such structure holds in a neighborhood of any point where at least one tangent cone has $(m-1)$-dimensional spine. These structural results are indeed the byproduct of a theorem that proves (for any modulus) uniqueness and decay towards such tangent cones. The underlying strategy of the proof is inspired by the techniques developed by Simon in a class of *multiplicity one* stationary varifolds. The major difficulty in our setting is produced by the fact that the cones and surfaces under investigation have arbitrary multiplicities ranging from 1 to $\lfloor \frac{p}{2} \rfloor$.

We want to mention that the presented result can as well derived by the theory for stable varifolds developed by N. Wickramasekera.

This talk is based on a joint work with C. De Lellis, A Marches, S. Stuvard and L. Spolaor.

Frobenius theorem for weak submanifolds

Annalisa Massaccesi

The question of producing a foliation of the n-dimensional Euclidean space with k-dimensional submanifolds which are tangent to a prescribed k-dimensional simple vectorfield is part of the celebrated Frobenius theorem: a decomposition in smooth submanifolds tangent to a given vectorfield is feasible (and then the vectorfield itself is said to be integrable) if and only if the vectorfield is involutive. In this seminar I will summarize the results obtained in collaboration with G. Alberti, A. Merlo and E. Stepanov when the smooth submanifolds are replaced by weaker objects, such as integral or normal currents or even contact sets with "some" boundary regularity. I will also provide Lusin-type counterexamples to the Frobenius property for rectifiable currents. Finally, I will try to highlight the connection between involutivity/integrability à la Frobenius and Carnot-Carathéodory spaces and how to apply our techniques in this framework.

Multiphase free discontinuity problems: monotonicity formula and regularity results

Dorin Bucur

We discuss regularity properties of local solutions to free discontinuity problems characterized by the presence of multiple phases. The main motivation comes from the analysis of the multiphase Mumford-Shah functional or from shape optimization problems with Robin boundary conditions. We prove that the jump set is essentially closed and Ahlfors regular. The proof relies on a multiphase monotonicity formula and on a sharp collective Sobolev extension result for functions with disjoint supports on a sphere. This is a joint work with I. Fragala and A. Giacomini.

A spectral shape optimization problem with a nonlocal competing term

Dario Mazzoleni

We study the minimization under a measure constraint of a functional made as the sum of a cohesive spectral term and a repulsive Riesz-type interaction functional, namely

$$\min\Big\{\lambda_1(\Omega) + \varepsilon \int_\Omega \int_\Omega \frac{1}{|x-y|^{N-\alpha}}\, dx\, dy : \Omega \subset \mathbb{R}^N, |\Omega| = 1\Big\},$$

where $\alpha \in (1, N)$, $\varepsilon > 0$ and λ_1 denotes the first eigenvalue of the Dirichlet Laplacian.

We show that there is a threshold $\varepsilon_1 > 0$ such that for all $\varepsilon \leq \varepsilon_1$ existence of minimizers occurs. Moreover we prove, by means of an expansion analysis, that the ball is a rigid minimizer. We also show that there is another threshold $\varepsilon_2 > \varepsilon_1$ such that if $\varepsilon \geq \varepsilon_2$, then minimizers do not exist (at least in a suitable class of admissible sets).

The techniques and tools needed in the proofs are very broad. We employ spectral quantitative inequalities, the regularity of free boundaries, spectral surgery arguments and shape variations.

This is a joint project with Berardo Ruffini (Bologna).

Quantitative Faber-Krahn inequalities and the ACF Monotonicity Formula

Robin Neumayer

Among all drum heads of a fixed area, a circular drum head produces the vibration of lowest frequency. The general dimensional analogue of this fact is the Faber-Krahn inequality, which states that balls have the smallest principal Dirichlet eigenvalue among subsets of Euclidean space with a fixed volume. I will discuss new quantitative stability results for the Faber-Krahn inequality on Euclidean space, the round sphere, and hyperbolic space, as well as an application to the Alt-Caffarelli-Friedman monotonicity formula used in free boundary problems. This is based on joint work with Mark Allen and Dennis Kriventsov.

Some results on harmonic maps with free boundary and beyond

Yannick Sire

The theory of harmonic maps with free boundary is an old topic in geometric analysis. I will report on recent results on their Ginzburg-Landau approximation, regularity theory, and their heat flow. I will also describe several models in the theory of liquid crystals where the heat flow of those maps appears, emphasizing on some well-posedness issues and some hints on the construction of blow-up solutions. Several important results in geometric analysis such as extremal metrics for the Steklov eigenvalues for instance make a crucial use of such maps. I?ll give some open problems and will try to explain how to attack few open questions in the field using tools recently developed.

Boundary regularity for higher multiplicity boundaries of 2D area minimizing currents

Simone Steinbrüchel

The Plateau Problem has been studied in different settings and many tools have been created such as integral currents, Almgren's frequency function and center manifold. They are used to analyze the interior and the boundary of minimizers in different dimensions and codimensions. In joint work with C. De Lellis and S. Nardulli, we take a first step into the regularity at the boundary if it has dimension 1, is taken with multiplicity greater than 1 and has a convex barrier. This question was raised by B. White and so far, if the codimension is greater than 1, nothing is known about it. In this talk, I will give an overview of the methods used in the regularity theory of integral currents and then explain how to adapt them to the higher multiplicity regime.

Free boundary problems in the spatial segregation of some competing systems

Nicola Soave

In this talk we present some results concerning the spatial segregation in systems with strong competition. In particular, we focus on systems of equations where the diffusion of each density is described by a different operator. We present an anisotropic Alt-Caffarelli-Friedman montonicity formula, and derive some Liouville-type theorems for subsolutions of these classes of systems. As a consequence, we prove uniform bounds in Hölder spaces for families of solutions, and discuss the asymptotic behavior in the limit of strong competition. The content of the talk is part of ongoing project with S. Terracini.

Boundary unique continuation of Dini domains

Zihui Zhao

Let u be a harmonic function in a domain $\Omega \subset \mathbb{R}^d$. It is known that in the interior, the singular set $\mathcal{S}(u) = \{u = |\nabla u| = 0\}$ is $(d-2)$-dimensional, and moreover $\mathcal{S}(u)$ is $(d-2)$-rectifiable and its Minkowski content is bounded (depending on the frequency of u). We prove the analogue near the boundary for C^1-Dini domains: If the harmonic function u vanishes on an open subset E of the boundary, then near E the singular set $\mathcal{S}(u) \cap \overline{\Omega}$ is $(d-2)$-rectifiable and has bounded Minkowski content. Dini domain is the optimal domain for which ∇u is continuous towards the boundary, and in particular every $C^{1,\alpha}$ domain is Dini. The main difficulty is the lack of the monotonicity formula for the frequency function near the boundary of a Dini domain. This is joint work with Carlos Kenig.

On the regularity of singular sets of minimizers for the Mumford-Shah energy

Matteo Focardi

We will survey on the regularity theory of minimizers of the Mumford-Shah functional, focusing in particular on that of the corresponding singular sets. Starting with nowadays classical results, we will finally discuss more recent developments.
