Regularity of the two-phase free boundaries

Bozhidar Velichkov

This is an extended version of the Oberwolfach report on a joint work with Guido De Philippis and Luca Spolaor. Many of the statements and the proofs, especially the ones regarding the one-phase problem, are intentionally oversimplified.

This talk is dedicated to the regularity of the free boundaries of the two-phase Bernoulli problem in any dimension. We first present some tools and ideas for the one-phase problem and then we discuss the two-phase case and our main result.

1. One-phase free boundaries

Let $D \subset \mathbb{R}^d$ be a fixed domain (for simplicity we suppose that D is the unit ball B_1), $\varphi : \partial D \to \mathbb{R}$ be a nonnegative function and $\Lambda > 0$ be a given constant. We consider the one-phase Bernoulli problem

$$(1) \qquad \textit{Minimize} \quad \int_D |\nabla u|^2\,dx + \Lambda|\{u>0\}\cap D| \quad \textit{among}$$
$$\textit{all functions} \quad u \in H^1(D) \quad \textit{such that} \quad u = \varphi \quad \textit{on} \quad \partial D.$$

1.1. **Example.** In dimension one, if D is the interval $[0,1]$, and if, for instance,

$$\phi(0) = a > 0 \qquad \text{and} \qquad \phi(1) = 0,$$

then it is easy to check that the minimizer should have the form

$$u(t) = \frac{a}{\ell}(\ell - t) \quad \text{for} \quad 0 \le t \le \ell\,, \qquad u(t) = 0 \quad \text{when} \quad t \ge \ell.$$

Now, a straightforward computation gives that u is optimal, when $\ell = 1$ or

$$|u'(\ell)| = \frac{a}{\ell} = \sqrt{\Lambda}.$$

In particular, one can notice that when Λ is large the solution is not regular as the gradient jumps from $\sqrt{\Lambda}$ to 0 where the function vanishes.

1.2. **Lipschitz continuity.** The optimal regularity for u (in any dimension) was obtained by Alt and Caffarelli. In [1] they showed that if $u : D \to \mathbb{R}$ is a solution of the one-phase problem (1), then it is (locally) Lipschitz continuous.[1]

[1] In the same paper Alt and Caffarelli prove a regularity result for the free boundary. We discuss a different strategy which was developed in several more recent papers by different authors. More details can be found in [6].

1.3. **First variation and its consequences.** Let ξ be a smooth vector field, compactly supported in D. Then, for small t, we can test the optimality of u with $u_t(x) := u(x - t\xi(x))$. Taking the derivative of the energy at $t = 0$, we get

$$0 = \int_D \Big(-2\nabla u\, D\xi\, (\nabla u)^t + \mathrm{div}\xi\, |\nabla u|^2 + \Lambda \mathrm{div}\, \xi \Big)\, dx.$$

We notice if u and $\partial\{u > 0\}$ are regular enough this is equivalent to

$$0 = \int_{\partial\{u>0\}} \big(-|\nabla u|^2 + \Lambda \big)\, (\xi \cdot \nu)\, d\mathcal{H}^{d-1},$$

where ν is the exterior normal to $\partial\{u > 0\}$. In particular, since this holds for an arbitrary ξ, we obtain that the optimality condition

$$|\nabla u| = \sqrt{\Lambda} \quad \text{on} \quad \partial\{u > 0\} \cap D, \tag{2}$$

should be satisfied (at least in some weak sense) also in dimension $d \geq 2$.

Another crucial consequence of this first order optimality condition is the Weiss' monotonicity formula [7], which is an important tool in the study of the blow-up limits. It also allows to apply a Federer dimension reduction principle and to give an the estimate of the dimension of the singular set.

1.4. **Blow-up limits.** The Lipschitz continuity of u implies that if $x_0 = 0$ is a point on the free boundary $\partial\{u > 0\} \cap D$, then the family of functions $u_r(x) = \frac{1}{r} u(rx)$ is (locally) uniformly Lipschitz. In particular, every sequence $r_n \to 0$ has a subsequence (still denoted by r_n) such that u_{r_n} converges locally uniformly to some $u_0 : \mathbb{R}^d \to \mathbb{R}$ (which a priori depends on the sequence r_n). Moreover:

- u_0 is a non-zero function (this non-degeneracy lemma can be found in [1]);
- u_0 is a local minimizer of the one-phase problem, that is,

$$\int_{B_R} |\nabla u_0|^2\, dx + |B_R \cap \{u_0 > 0\}| \leq \int_{B_R} |\nabla v|^2\, dx + |B_R \cap \{v > 0\}|,$$

  for every ball B_R and every $v \in H^1(B_R)$ with $v = u_0$ on ∂B_R.
- u_0 is 1-homogeneous (as a consequence of the monotonicity formula).

We can use this information to give sense to the optimality condition (2).

1.5. **Viscosity solutions.** Let u be a solution to (1). Then u satisfies (2) in the following (viscosity) sense. For every $x_0 \in \partial\{u > 0\} \cap D$ and $\varphi \in C^\infty(D)$,

- if $u(x_0) = \varphi(x_0)$ and $u \leq \varphi_+$ in D, then $|\nabla\varphi(x_0)| \geq \sqrt{\Lambda}$;
- if $u(x_0) = \varphi(x_0)$ and $u \geq \varphi_+$ in D, then $|\nabla\varphi(x_0)| \leq \sqrt{\Lambda}$.

Let us prove the first point. Let $x_0 = 0$ and $r_n \to 0$ be such that the sequence u_{r_n} converges to some blow-up limit u_0. Since φ is regular, $\varphi_{r_n}(x) := \frac{1}{r_n}\varphi(r_n x)$ converges to the linear function $x \mapsto x \cdot \nabla\varphi(0)$. Then $u_0(x) \leq \big(x \cdot \nabla\varphi(0)\big)_+$. Using that u_0 is a non-zero 1-homogeneous function which is harmonic in its support (which is contained in the half-space $\big\{x : x \cdot \nabla\varphi(0) > 0\big\}$), we get that

$$u_0(x) = C\big(x \cdot \nabla\varphi(0)\big)_+ \quad \text{for some constant} \quad 0 < C \leq 1.$$

Now, since u_0 and $\partial\{u_0 > 0\}$ are regular, (2) holds for u_0 in the classical sense. Thus, C should be equal to $|\nabla\varphi(0)|^{-1}\sqrt{\Lambda}$. This proves that $|\nabla\varphi(0)| \geq \sqrt{\Lambda}$. The case when φ touches u from below is similar.

1.6. **Regularity of the flat one-phase free boundaries.** In [4] De Silva proved the following theorem.

Theorem (De Silva [4]). *There are $\varepsilon_0 > 0$ and $0 < \rho < 1$ such that if u is Lipschitz continuous, harmonic in $\{u > 0\}$, satisfies (2) in viscosity sense and*

$$\sqrt{\Lambda}\,(x_d - \varepsilon)_+ \leq u(x) \leq \sqrt{\Lambda}\,(x_d + \varepsilon)_+ \quad \textit{for every} \quad x \in B_1$$

for some $\varepsilon < \varepsilon_0$, then there is a unit vector ν, ε-close to e_d and such that

$$\sqrt{\Lambda}\,(x \cdot \nu - \varepsilon/2)_+ \leq \frac{1}{\rho}u(\rho x) \leq \sqrt{\Lambda}\,(x \cdot \nu + \varepsilon/2)_+ \quad \textit{for every} \quad x \in B_1.$$

As a consequence, a classical argument gives that:

Corollary. *Let u be a solution of (1). If at some point $x_0 \in \partial\{u > 0\} \cap D$ the function u has a blow-up limit of the form $u_0(x) = \sqrt{\Lambda}\,x_d^+$, then $\partial\{u > 0\}$ is $C^{1,\alpha}$ manifold in a neighborhood of x_0.*

The proof of the De Silva's theorem can be divided into two main steps.

Step 1. The first step is to prove the non-rescaled version of the theorem:

Lemma (De Silva [4]). *In the hypotheses of the above theorem, if*

$$\sqrt{\Lambda}\,(x_d + A)_+ \leq u(x) \leq \sqrt{\Lambda}\,(x_d + B)_+ \quad \textit{for every} \quad x \in B_1$$

for some $0 < B - A < \varepsilon_0$, then there are a, b such that $0 < b - a < \frac{1}{2}(B - A)$ and

$$\sqrt{\Lambda}\,(x_d + a)_+ \leq u(x) \leq \sqrt{\Lambda}\,(x_d + b)_+ \quad \textit{for every} \quad x \in B_\rho.$$

Notice that this statement can be summarized in the following claim:

(3) *If u is ε-close to a solution of the form $\sqrt{\Lambda}\big((x - x_0) \cdot \nu\big)_+$ in B_1, then*
u is $\varepsilon/2$-close to a solution of the form $\sqrt{\Lambda}\big((x - y_0) \cdot \nu\big)_+$ in B_ρ

(the point y_0 might be different from x_0 but the direction ν remains the same).

Proof. If u, calculated in some fixed point (say $\bar{x} = 1/5e_d$), is bigger that $x_d^+\sqrt{\Lambda}$ calculated in the same point, then the lower bound on u can be improved in B_ρ. In fact, in [4] it was constructed an increasing family of functions w_t such that

$$\Delta w_t > 0, \quad |\nabla w_t| > \sqrt{\Lambda}, \quad w_t = u \text{ on } \partial B_1, \quad \text{and} \quad w_t \leq u \text{ at } \bar{x},$$

for every $0 < t < 1$. Since none of w_t can touch u from below, we get that $w_1 \leq u$. This provides the improvement in B_ρ. □

Step 2. Let ρ be fixed. Suppose by contradiction that such an ε_0 does not exist. Then, there is a sequence u_n of ε_n-flat solutions which are not flatter in B_1. But then, Step 1, gives that the sequence

$$v_n(x) = \frac{u_n(x) - \sqrt{\Lambda}x_d^+}{\varepsilon_n}$$

converges in some suitable sense to a function v. One can prove (see [4]) that the limit v is a solution to some *limit problem.* In the one-phase case v is harmonic in $\{x_d > 0\}$ with Neumann boundary condition on $\{x_d = 0\}$. Now the classical regularity of the harmonic functions gives that, for a dimensional constant $C_d > 0$,

$$x \cdot \nabla v(0) - C_d\rho^2 \le v(x) \le x \cdot \nabla v(0) + C_d\rho^2 \quad \text{in} \quad B_\rho.$$

Using the convergence of v_n to v, we get that u_n is flatter in the direction of the vector $e_d + \varepsilon_n \nabla v(0)$. This is a contradiction.

2. Two-phase free boundaries - the main result

Given a domain D in $\mathbb{R}^d$, a (sign-changing) function $\varphi : \partial D \to \mathbb{R}$ and constants $\Lambda_+ > 0$ and $\Lambda_- > 0$, we consider the following two-phase Bernoulli problem:

$$\text{(4)} \quad \textit{Minimize} \quad \int_D |\nabla u|^2\,dx + \Lambda_+|\{u>0\}\cap D| + \Lambda_-|\{u<0\}\cap D|$$
$$\textit{among all functions} \quad u \in H^1(D) \quad \textit{such that} \quad u = \varphi \quad \textit{on} \quad \partial D.$$

This problem was introduced by Alt, Caffarelli and Friedman in [2]. The Lipschitz continuity of the solutions was also proved in [2]. A Weiss-type monotonicity formula can be obtained exactly as for the one-phase problem [7] and, as in the one-phase case, it implies that all the blow-up limits are one-homogeneous functions. Finally, reasoning as in Section 1.5, we get that if x_0 is a two-phase point

$$x_0 \in \partial\{u>0\} \cap \partial\{u<0\} \cap D,$$

then every blow-up limit of u at x_0 is of the form

$$u_0(x) = \alpha(x\cdot\nu)_+ - \beta(x\cdot\nu)_-, \tag{5}$$

where ν is a unit vector (that might depend on the blow-up sequence) and α, β are positive constants (one can show that α and β depend only on x_0 and not on the blow-up sequence) such that

$$\alpha^2 \ge \Lambda_+, \quad \beta^2 \ge \Lambda_-, \quad \alpha^2 - \beta^2 = \Lambda_+ - \Lambda_-.$$

One can express this as an optimality condition in viscosity sense, precisely as in the one-phase case (see [3]).

In [3], with De Philippis and Spolaor, we proved the following theorem about the regularity of the free boundary around two-phase points.

Theorem (De Philippis-Spolaor-V. [3])**.** *Let u be a solution of* (4)*. Then, in a neighborhood of every two-phase point $x_0 \in \partial\{u>0\}\cap\partial\{u<0\}\cap D$, both free boundaries $\partial\{u>0\}$ and $\partial\{u<0\}$ are $C^{1,\alpha}$ manifolds (in any dimension $d \ge 2$).*

Remark 1. The same regularity result holds for viscosity solutions, which are ε close to a solution of the form (5), precisely as in the De Silva's Theorem.

Remark 2. In dimension two, this theorem was proved in our earlier paper [5] via an epiperimetric inequality.

The proof of the above theorem follows the main steps from the proof of the De Silva's Theorem, but there are two main differences. The first one is technical and comes from the fact that the limit problem from Step 2 is a thin two-membrane problem, but this does not require a change in the general approach to the problem. The second difference is hidden in Step 1. In fact the statement (3) turns out to be false in the two-phase case. Even in dimension one. For instance, for every $\varepsilon > 0$, the function

$$u_\varepsilon(t) = \sqrt{\Lambda_+}(t+\varepsilon)_+ - \sqrt{\Lambda_-}(t-\varepsilon)_-$$

is a solution to the two-phase problem and is ε-close, in the interval $(-1, 1)$, to the 1-homogeneous global solution

$$\sqrt{\Lambda_+}t_+ - \sqrt{\Lambda_-}t_-.$$

but this closeness cannot be improved in the smaller interval $(-\rho, \rho)$.

Acknowledgements. The author warmly thanks the Oberwolfach reserach center for the hospitality during the 2020 Calculus of Variations meeting. The work of BV is partially supported by the European Research Council (ERC), under the European Union's Horizon 2020 research and innovation programme, through the project ERC VAREG - *Variational approach to the regularity of the free boundaries* (grant agreement No. 853404).

References

[1] H.W. Alt, L.A. Caffarelli. *Existence and regularity for a minimum problem with free boundary.* J. Reine Angew. Math. **325** (1981), 105–144.

[2] H.W. Alt, L.A. Caffarelli, A. Friedman. *Variational problems with two phases and their free boundaries.* Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461.

[3] G. De Philippis, L. Spolaor, B. Velichkov, *Regularity of the free boundary for the two-phase Bernoulli problem*, Preprint (2019).

[4] D. De Silva. *Free boundary regularity from a problem with right hand side.* Interfaces and Free Boundaries **13** (2) (2011), 223–238.

[5] L. Spolaor, B. Velichkov, *An epiperimetric inequality for the regularity of some free boundary problems: the 2-dimensional case*, Comm. Pure Appl. Math. **72** (2) (2018), 375–421.

[6] B. Velichkov. *Regularity of the one-phase free boundaries.* Lecture notes available at `http://cvgmt.sns.it/paper/4367/`

[7] G. S. Weiss. *Partial regularity for a minimum problem with free boundary.* J. Geom. Anal. **9** (2) (1999), 317–326.