Regularity of the optimal sets for the second Dirichlet eigenvalue

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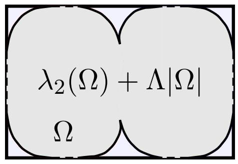
Summary

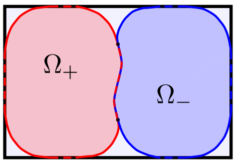


Setting. Given a bounded open set $\Omega \subset \mathbb{R}^d$, we consider the functional

$$\Omega \mapsto \lambda_2(\Omega) + \Lambda\,\mathrm{Vol}(\Omega) \qquad \text{(TP)}$$

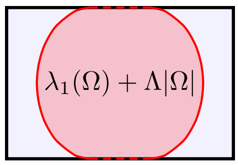
where $\Lambda > 0$ is a constant, $\mathrm{Vol}(\Omega)$ is the Lebesgue measure of Ω, and $\lambda_2(\Omega)$ is the second eigenvalue of the Dirichlet Laplacian on Ω.

Main result. We fix a $C^{1,\alpha}$-regular bounded open set $D \subset \mathbb{R}^d$ and we prove a regularity result for the sets Ω that minimize $\lambda_2 + \Lambda\mathrm{Vol}$ among all open subset of D. Precisely, we prove that if Ω is a minimizer of $\lambda_2 + \Lambda\mathrm{Vol}$, then it contains two disjoint open sets Ω_+ and Ω_- such that:

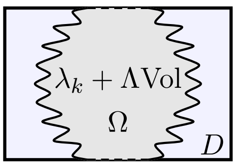


- $\Omega_+ \cup \Omega_-$ is a minimizer of $\lambda_2 + \Lambda\mathrm{Vol}$;
- the boundaries $\partial\Omega_+$ and $\partial\Omega_-$ are $C^{1,\alpha}$ regular in a neighborhood of $\partial\Omega_+ \cap \partial\Omega_-$;
- the free boundaries $\partial\Omega_\pm$ are $C^{1,\alpha}$ regular in a neighborhood of $\partial\Omega_\pm \cap \partial D$;
- the one-phase free boundaries $\partial\Omega_+ \setminus (\partial D \cup \partial\Omega_-)$ and $\partial\Omega_- \setminus (\partial D \cup \partial\Omega_+)$ are $C^{1,\alpha}$ regular up to a closed $(d-5)$-dimensional set.

In particular, in dimension $d \leq 4$, the sets Ω_+ and Ω_- are $C^{1,\alpha}$ regular.



History. Shape optimization problems for functionals involving the Lebesgue measure and the eigenvalues of the Dirichlet Laplacian were studied for more than a century. The first regularity result for such functionals was proved in 2009 by Briançon and Lamboley [BL] for $\lambda_1 + \Lambda\mathrm{Vol}$. The analysis was based on the technique of Alt and Caffarelli and was recently improved in [RTV].



For functionals involving higher eigenvalues, as for instance $\lambda_k + \Lambda\mathrm{Vol}$, the existence of an optimal set (in the large class of quasi-open sets) was proved by Buttazzo and Dal Maso [BDM]. Little is known about the regularity and the structure of the optimal sets. The first result in this direction was proved in [MTV] (and improved in [KL1]) for the functional $\lambda_1 + \lambda_2 + \cdots + \lambda_k + \Lambda\mathrm{Vol}$. When the functional does not involve the first eigenvalue λ_1, the regularity of the optimal sets is more involved. Indeed, the only regularity result for the free boundaries of the minimizers of $\lambda_k + \Lambda\mathrm{Vol}$ (for general $k \geq 2$) was proved by Kriventsov and Lin in [KL2], where they show that the flat free boundaries are smooth. In the case $k = 2$, this corresponds to the regularity of $\partial\Omega_\pm \setminus (\partial D \cup \partial\Omega_\mp)$.

In particular, our result is the first that gives a complete account on the regularity of the optimal sets for functionals of the form $\lambda_k + \Lambda\mathrm{Vol}$, for $k \geq 2$.

Strategy of the proof. We first show that we can rewrite the shape optimization problem for $\lambda_2 + \Lambda\mathrm{Vol}$ as a two-phase free boundary problem for the (non-differentiable) functional

$$J(u) = \max\left\{\int_{\{u>0\}} |\nabla u|^2\,dx\ ;\ \int_{\{u<0\}} |\nabla u|^2\,dx\right\} + \Lambda\mathrm{Vol}\big(\{u \neq 0\}\big) \qquad \text{where} \quad u \in H^1_0(D)\,.$$

Then, we use a three-phase monotonicity formula as in [BuV] and the result from [RTV] to describe the free boundary in a neighborhood of ∂D. The main part of the paper is dedicated to prove that the minimizer u of J are viscosity solutions of a two-phase free boundary problem; once we have this, the regularity of the two-phase free boundary $\partial\Omega_+ \cap \partial\Omega_-$ follows from [DSV].

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