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Regularity of the free boundary in the obstacle problem

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Contents

Introduction

The obstacle problem is a classic problem in mathematical analysis and the calculus of variations, with applications in various fields of science and engineering, ranging from material theory to mathematical finance. Intuitively, the problem can be described as the search for the optimal configuration of an elastic membrane subject to a physical obstacle. More formally, it involves finding a function that minimizes a certain energy functional while satisfying a set of constraints imposed by the obstacle itself.

One of the most fascinating aspects of the obstacle problem is the nature of the free boundary, which is the boundary between the region where the solution touches the obstacle and the region where it remains free. The regularity of this free boundary is crucial for understanding the fine structure of the solutions to the problem.

In this thesis, we aim to explore the obstacle problem with a particular focus on the regularity of the free boundary. After reviewing the classical theory of the obstacle problem, we will focus on modern analytical techniques, such as the monotonicity formulas methods and geometric analysis tools, to establish optimal regularity results.

In particular, in Chapter 2, we will introduce the classical formulation of the obstacle problem and study the fundamental properties of the solutions. In Chapter 3, we will focus on the study of the free boundary and its regularity, with particular attention to the regular points, while in Chapter 4, we will examine the singular points. Finally, Chapter 5 discusses Schaeffer's conjecture in two dimensions, along with the details of Monneau's proof.

The core material for this thesis, in addition to the articles cited in the bibliography, is based on the book "Regularity Theory for Elliptic PDE" by Xavier Fernández-Real and Xavier Ros-Oton.

Chapter 1

Preliminaries

We next give a quick review of some basic definitions about L^p, Sobolev, and Hölder spaces, and some results that will be used later in the thesis.

1.1 Sobolev and Hölder spaces

L^p **spaces**. Given $\Omega \subset \mathbb{R}^n$ and $1 \leq p < \infty$, the space $L^p(\Omega)$ is the set

$$L^p(\Omega) := \left\{ u \text{ measurable in } \Omega \; : \int_\Omega |u|^p dx < \infty \right\}.$$

It is a Banach space, with the usual norm $||u||_{L^p(\Omega)} := (\int_\Omega |u|^p dx)^{1/p}$.
When $p = \infty$, the space $L^\infty(\Omega)$ is the set of bounded functions (up to sets of measure zero), with the norm $||u||_{L^\infty(\Omega)} := \operatorname{ess\,sup}_\Omega |u|$.

Theorem 1.1. *If $u \in L^1(\Omega)$ then for almost every $x \in \Omega$ we have*

$$\lim_{r \to 0} ⨍_{B_r(x)} |u(x) - u(y)| dy = 0.$$

When this holds at a point $x \in \Omega$, we say that x is a Lebesgue point of u.

Here, and throughout the thesis, $⨍_\Omega$ denotes the average $\frac{1}{|\Omega|} \int_\Omega$, where $\Omega \subset \mathbb{R}^n$ is any set of finite positive measure.

Corollary 1.2. *If $u \in L^1(\Omega)$, and*

$$\int_\Omega u(x)v(x)\, dx = 0 \quad \textit{for all } v \in C_c^\infty(\Omega).$$

Then, $u = 0$ a.e in Ω.

Integration by parts A fundamental identity in the study of PDEs is the following.

Theorem 1.3 (Integration by parts). *Assume $\Omega \subset \mathbb{R}^n$ is any bounded C^1 domain. Then, for any u, $v \in C^1(\bar{\Omega})$ we have*

$$\int_\Omega \partial_i u \, v \, dx = - \int_\Omega u \, \partial_i v \, dx + \int_{\partial\Omega} uv \, \nu_i \, dS, \tag{1.1}$$

where ν is the unit (outward) normal vector to $\partial\Omega$, and $i = 1, \dots, n$.

Notice that, as an immediate consequence, we find the divergence theorem, as well as Green's first identity:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = - \int_\Omega u \, \Delta v \, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, dS.$$

The regularity requirements of Theorem 1.3 can be relaxed. Indeed the domain Ω need only to be Lipschitz, while only u, $v \in H^1(\Omega)$ is necessary in (1.1) - where $H^1(\Omega)$ is a Sobolev space, defined below.

Sobolev spaces. Given any domain $\Omega \subset \mathbb{R}^n$ and $1 \le p \le \infty$, the Sobolev spaces $W^{1,p}(\Omega)$ consist of all functions whose (weak) derivatives are in $L^p(\Omega)$, namely

$$W^{1,p}(\Omega) := \Big\{ u \in L^p(\Omega) \ : \ \partial_i u \in L^p(\Omega) \text{ for } i = 1, \dots, n \Big\}.$$

We refer to the books [evans] [Bre] inserire bibliografia! for the definition of weak derivatives and a detailed exposition on Sobolev spaces.

- **(S1)** The spaces $W^{1,p}(\Omega)$ are complete.
- **(S2)** The inclusion $W^{1,p}(\Omega) \subset L^p(\Omega)$ is compact,
- **(S3)** The space $H^1(\Omega) := W^{1,2}(\Omega)$ is a Hilbert space withe the scalar product

$$(u, v)_{H^1(\Omega)} := \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.$$

- **(S4)** Any bounded sequence $\{u_k\}$ in the Hilbert space $H^1(\Omega)$ contains a weakly convergent sequence $\{u_{k_j}\}$, that is, there exists $u \in H^1(\Omega)$ such that

$$(u_{k_j}, v)_{H^1(\Omega)} \to (u, v)_{H^1(\Omega)} \ \text{ for all } v \in H^1(\Omega). \tag{1.2}$$

in addition, such u will satisfy

$$||u||_{H^1(\Omega)} \le \liminf_{j \to \infty} ||u_{k_j}||_{H^1(\Omega)} \tag{1.3}$$

and since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ one has

$$||u||_{L^2(\Omega)} = \lim_{j \to \infty} ||u_{k_j}||_{L^2(\Omega)}. \tag{1.4}$$

- **(S5)** Let Ω be any bounded Lipschitz domain, and $1 \le p \le \infty$. Then, there is a continuous (and compact for $p > 1$) trace operator from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$. For C^0 functions, such trace operator is simply $u \mapsto u|_{\partial\Omega}$.
  Because of this, for any function $u \in H^1(\Omega)$ we will still denote by $u|_{\partial\Omega}$ its trace on $\partial\Omega$.

- **(S6)** For $1 \le p < \infty$ $C^\infty(\Omega)$ functions are dense in $W^{1,p}(\Omega)$. Moreover, if Ω is bounded and Lipschitz, $C^\infty(\bar{\Omega})$ are dense in $W^{1,p}(\Omega)$.

- **(S7)** For $1 \le p < \infty$, we define the space $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $(W^{1,p}(\Omega)$. Similarly, we denote $H_0^1(\Omega) := W_0^{1,p}(\Omega)$. When Ω is bounded and Lipschitz, it is the space of functions $u \in W^{1,p}(\Omega)$ such that $u|_{\partial\Omega} = 0$.

- **(S8)** if $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$, then for any subdomain $K \subset\subset \Omega$ we have

$$\left|\left|\frac{u(x+h)-u(x)}{|h|}\right|\right|_{L^p(K)} \le C||\nabla u||_{L^p(\Omega)}$$

  for all $h \in B_\delta$, with $\delta > 0$ small enough. Conversely, if $u \in L^p(\Omega)$, $1 < p \le \infty$, and

$$\left|\left|\frac{u(x+h)-u(x)}{|h|}\right|\right|_{L^p(K)} \le C$$

  for every $h \in B_\delta$, then $u \in W^{1,p}(\Omega)$ and $||\nabla u||_{L^p(\Omega)} \le C$. (This property fails when $p = 1$.)

Theorem 1.4 (Sobolev inequality)**.** *If $p < n$, then*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{1/p^*} \le C\left(\int_{\mathbb{R}^n} |\nabla|^p\, dx\right)^{1/p}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

for some constant C depending only on n and p. In particular, we have a continuous inclusion $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$.

Notice that if $p \uparrow n$ we have $p^* \to \infty$. In the limit case $p = n$, however, it is not true that $W^{1,n}$ functins are bounded.

Theorem 1.5 (Morrey inequality)**.** *If $p > n$, then*

$$\sup_{x \ne y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C\left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{1/p}, \quad \alpha = 1 - \frac{n}{p},$$

for some constant C depending only on n and p.

In particular, when $p > n$ any function in $W^{1,p}(\Omega)$ is continuous (in the sense that it admits a continuous equivalent function).

Theorem 1.6 (Poincaré inequality)**.** *. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $p \in [1, \infty)$. Then, for any $u \in W^{1,p}(\Omega)$ we have*

$$\int_\Omega |u - u_\Omega|^p \, dx \le C_{\Omega,p} \int_\Omega |\nabla u|^p \, dx, \quad u_\Omega := \fint u,$$

and

$$\int_\Omega |u|^p \, dx \le C_{\Omega,p} \left(\int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} |u|_{\partial\Omega}|^p d\sigma \right).$$

The constants depend only on n, p and Ω.

Hölder spaces. Given $\alpha \in (0,1)$, The Hölder space $C^{0,\alpha}(\bar{\Omega})$ is the set of continuous functions $u \in C(\bar{\Omega})$ such that the Hölder semi-norm is finite,

$$[u]_{C^{0,\alpha}(\bar{\Omega})} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

The Hölder norm is

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} := \|u\|_{L^\infty(\Omega)} + [D^k u]_{C^{0,\alpha}(\bar{\Omega})},$$

where

$$\|u\|_{C^k(\bar{\Omega})} := \sum_{j=1}^{k} \|D^j u\|_{L^\infty(\Omega)}.$$

Notice that this yields the inclusions

$$C^0 \supset C^{0,\alpha} \supset \mathrm{Lip} \supset C^1 \supset C^{1,\alpha} \supset \cdots \supset C^\infty.$$

We will write $\|u\|_{C^{k,\alpha}(\Omega)}$ instead of $\|u\|_{C^{k,\alpha}(\bar{\Omega})}$.

We state now one of the most basic theorem for the study of harmonic functions. It gives a kind of "maximum principle in quantitative form". We will write that $u \in H^1$ is harmonic, meaning in the weak sense. It is well known that as soon as a function is harmonic, it is immediately C^∞.

Theorem 1.7 (Harnack's inequality)**.** *Assume $u \in H^1(B_1)$ is a non-negative, harmnic function in B_1. Then the infimum and the supremum of u are comparable in $B_{1/2}$. That is,*

$$\begin{cases} \Delta u &= \text{ in } B_1 \\ u &= \text{ in } B_1 \end{cases} \implies \sup_{B_{1/2}} u \le C \inf_{B_{1/2}} u$$

for some constant C depending only on n.

Proof. This can be proved by the mean value property. Alternatively, we can use the Poisson kernel representation,

$$u(x) = c_n \int_{\partial B_1} \frac{(1-|x|^2)u(z)}{|x-z|^n} dz.$$

Notice that, for any $x \in B_{1/2}$ and $z \in \partial B_1$, we have $2^{-n} \leq |x-z|^n \leq (3/2)^n$ and $3/4 \leq 1-|x|^2 \leq 1$. Thus, since $u \geq 0$ in B_1,

$$C^{-1} \int_{\partial B_1} u(z)dz \leq u(x) \leq C \int_{\partial B_1} u(z)dz, \quad \text{for all } x \in B_{1/2},$$

for some dimensional constant C. In particular, for any $x_1,\ x_2 \in B_{1/2}$ we have that $u(x_1) \leq C^2 u(x_2)$. Taking the infimum for $x_2 \in B_{1/2}$ and the supremum for $x_1 \in B_{1/2}$, we reach that $\sup_{1/2} \leq \tilde{C} \inf_{B_{1/2}}$, for some dimensional constant $\tilde{C}$, as desired. □

Remark 1.8. There is nothing special about $B_{1/2}$. We can obtain a similar inequality in B_ϱ, with $\varrho < 1$, but the constant C would depend on ϱ as well. Indeed, repeating the previous argument, one gets that if $\Delta u = 0$ and $u \geq 0$ in B_1, then

$$\sup_{B_\varrho} u \leq \frac{C}{(1-\varrho)^n} \inf_{B_\varrho} u, \tag{1.5}$$

For some C depending only ib n, and $\varrho \in (0,1)$.

Lemma 1.9 (Hopf Lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be any domain satisfying the interior ball condition. Let $u \in C(\overline{\Omega})$ be any positive harmonic function in $\Omega \cap B_2$, with $u \geq 0$ on $\partial\Omega \cap B_2$.*

Then, $u \geq c_\circ d$ in $\Omega \cap B_1$ for some $c_\circ > 0$, where $d(x) := \operatorname{dist}(x, \Omega^c)$.

Proof. Since u is positive and continuous in $\Omega \cap B_2$, we have that $u \geq c_1 > 0$ in $\{d \geq \rho_\circ/2\} \cap B_{3/2}$ for some $c_1 > 0$.

Let us consider the solution of $\Delta w = 0$ in $B_{\rho_\circ} \setminus B_{\rho_\circ/2}$, with $w = 0$ on $\partial B_{\rho_\circ}$ and $w = 1$ on $\partial B_{\rho_\circ/2}$. In particular, it is immediate to check that $w \geq c_2(\rho_\circ - |x|)$ in $B_{\rho_\circ}$ for some $c_2 > 0$.

By using the function $c_1 w(x_\circ + x)$ as a subsolution in any ball $B_{\rho_\circ}(x_\circ) \subset \Omega \cap B_{3/2}$, we deduce that $u(x) \geq c_1 w(x_\circ + x) \geq c_1 c_2(\rho_\circ - |x - x_\circ|) \geq c_1 c_2 d$ in $B_{\rho_\circ}(x_\circ)$. Setting $c_\circ = c_1 c_2$ and using the previous inequality for every ball $B_{\rho_\circ}(x_\circ) \subset \Omega \cap B_{3/2}$, the result follows. □

Chapter 2

The obstacle problem

We now focus our attention to on a third type of nonlinear elliptic PDE: a free boundary problem. In this kind of problem we are no longer interested in the regularity of a solution u, but also in the study of an a priori unknown interphase Γ (the free boundary).

There is a wide variety of problems in applied sciences that can be described by PDEs that exhibit free boundaries. Many of such problems can be written as variational inequalities, for which the solution is obtained by minimizing a constrained energy functional. One of the most classical example is the obstacle problem.

Given a smooth function ϕ, the obstacle problem is the following:

$$\text{minimize} \quad \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx \quad \text{among all functions } v \geq \phi. \tag{2.1}$$

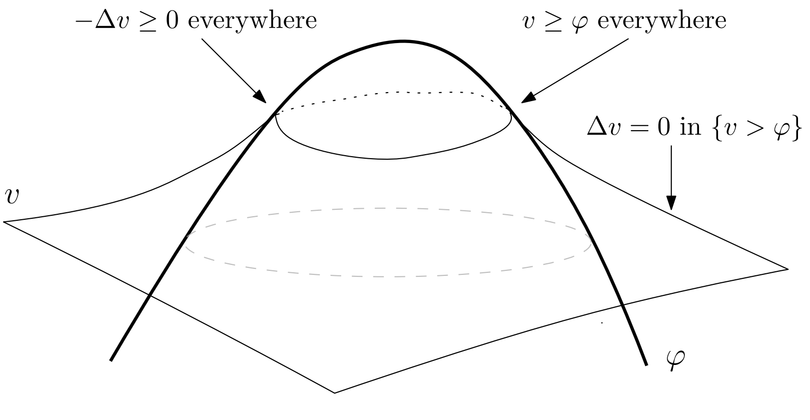

Figure 2.1: The function v minimizes the Dirichlet energy among all functions with the same boundary values situated above the obstacle.

The interpretation of such problem is clear: one looks for the least energy function v, but the set of admissible functions consists only of functions that are above a certain "obstacle" ϕ.

In the two-dimensional case, one can think of the solution v as a "membrane" which is elastic and is constrained to be above ϕ,(see Figure 2.1). The Euler-Lagrange equation of the minimization problem is the following:

$$\begin{cases} v & \geq \phi \text{ in } \Omega \\ \Delta v & \leq 0 \text{ in } \Omega \\ \Delta v & = 0 \text{ in the set } \{v > \phi\}, \end{cases} \tag{2.2}$$

with the boundary condition $v|_{\partial\Omega} = g$.

Indeed, notice that if we denote $\mathcal{F}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx$, then we will have

$$\mathcal{F}(v + \varepsilon\eta) \geq \mathcal{F}(v) \text{ for every } \varepsilon \geq 0, \text{ and } \eta \geq 0, \eta \in C_c^\infty(\Omega),$$

which yields $\Delta v \geq 0$ in Ω. That is, we can perturb v with nonnegative functions $(\varepsilon\eta)$ and we always get admissible functions $(v+\varepsilon\eta)$. However, due to the constraint $v \geq \phi$, we cannot perturb v with negative functions in all of Ω, but only in the set $\{v > \phi\}$. This is why we get $\Delta v \leq 0$ everywhere in Ω, but $\Delta v = 0$ only in $\{v > \phi\}$.

As we can see later, any minimizer of (2.1) is continuous, hence the set $\{v > \phi\}$ is open.)

Alternatively, we may consider $u := v - \phi$, and the problem is equivalent to

$$\begin{cases} u & \geq 0 \text{ in } \Omega \\ \Delta u & \leq f \text{ in } \Omega \\ \Delta u & = f \text{ in the set } \{u > 0\}, \end{cases} \tag{2.3}$$

where $f = \Delta\phi$.
Such solution u can be obtained as follows:

$$\text{minimize} \quad \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + fu \right\} dx \quad \text{among all functions } u \geq 0 \tag{2.4}$$

Indeed

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla(v-\phi)|^2 \, dx &= \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx \; + \; \frac{1}{2}\int_\Omega |\nabla\phi|^2 \; - \; \int_\Omega \nabla u \cdot \nabla\phi \, dx \\ &= \mathcal{F}(v) \; + \; \mathcal{F}(\phi) \; + \; \int_\Omega u\Delta\phi \, dx \; - \; \int_{\partial\Omega} g\frac{\partial\phi}{\partial\nu} \, dx, \end{aligned}$$

where $\mathcal{F}(\phi)$ and the boundary term are constant, so the variational problems (2.1) and (2.4) are equivalent. In other words, we can make the obstacle just zero, by adding a right-hand side f. Here, the minimization is subject to the boundary conditions $u|_{\partial\Omega} = \tilde{g} := g - \phi$.

The free boundary. Let us take a closer look at the obstacle (2.3).

One of the most important features of such problems is that it has *two* unknowns: the *solution* u, and the *contact set* $\{u = 0\}$. In other words, there are two regions in Ω: one in which $u = 0$; and one in which $\Delta u = f$.
These regions are characterized by the minimization problem (2.4). Moreover, if we denot

$$\Gamma := \partial\{u > 0\} \cap \Omega,$$

then this is called the *free boundary*.
The obstacle problem is a *free boundary problem*, as it involves an *unknown interface* Γ as part of the problem. More over is not difficult to see that the fact that u is a nonnegative supersolution must imply $\nabla u = 0$ on Γ, that is , we will have that $u \geq 0$ solves

$$\begin{cases} \Delta u &= f \text{ in } \{u > 0\} \\ u &= 0 \text{ on } \Gamma \\ \nabla u &= 0 \text{ on } \Gamma. \end{cases}$$

This is just an alternative way to write the Euler-Lagrange equation of the problem. In this way, the interface Γ appears clearly, and we see that we have both Dirichlet and Neumann conditions on Γ.

2.1 Basic properties of Solutions I

We proceed now to study the basic properties of solutions to the obstacle problem: existence of solutions, optimal regularity, and nondegeneracy.

Existence of solutions. Existence and uniqueness of solutions follows easily from the fact that the functional $\int_\Omega |\nabla v|^2 dx$ is convex, and that we want to minimize it in the closed convex set $\{v \in H^1(\Omega) \ : \ v \geq \phi\}$

Proposition 2.1 (Existence and uniqueness.). *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, abd let $g : \partial\Omega \to \mathbb{R}$ and $\phi \in H^1(\Omega)$ be such that*

$$\mathcal{C} := \{w \in H^1(\Omega) \ : \ w \geq \phi \text{ in } \Omega, \ w|_{\partial\Omega} = g\} \neq \varnothing.$$

Thene, there exists a unique minimizer of $\int_\Omega |\nabla v|^2 \, dx$ among all functions $v \in H^1(\Omega)$ satisfying $v \geq \phi$ in Ω and $v|_{\partial\Omega} = g$.

Proof. Let

$$I := \inf \left\{ \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx \ : \ w \in H^1(\Omega), \ w|_{\partial\Omega} = g, \ w \geq \phi \text{ in } \Omega \right\},$$

that is, the infimum value of $\mathcal{F}(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx$ among all admissible functions w. Let us take a sequence of functions $\{u_k\}$ such that

- $u_k \in H^1(\Omega)$.
- $u_k|_{\partial\Omega} = g$ and $u_k \geq \phi$ in Ω.
- $\mathcal{F}(u_k) \to I$ as $k \to \infty$.

Thanks to Poincaré inequality, the sequence $\{u_k\}$ is uniformly bounded in $H^1(\Omega)$, and therefore there exists a subsequence $\{u_{k_j}\}$ that converges to a certain function v strongly in L^2 and weakly in $H^1(\Omega)$. Moreover, by compactness of the trace operator we will have that $u_{k_j}|_{\partial\Omega} \to v|_{\partial\Omega}$ in $L^2(\partial\Omega)$, so that $v|_{\partial\Omega} = g$. Furthermore, such function v will satisfy $\mathcal{F}(v) \leq \liminf_{j\to\infty} \mathcal{F}(u_{k_j})$, and therefore it will be a minimizer of the energy functional. Since $u_{k_j} \geq \phi$ in Ω and $u_{k_j} \to v$ in $L^2(\Omega)$, we have $v \geq \phi$ in Ω. Thus, we have proved the existence of a minimizer v. Uniqueness follows directly from the strict convexity of the functional. Indeed if v is a solution for the obstacle problem then for every $u \in H_0^1(\Omega)$ we have

$$\begin{aligned}\mathcal{F}(v+u) &= \frac{1}{2}\int_\Omega |\nabla(v+u)|^2\,dx \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega \nabla v\cdot\nabla u\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx \\ &= \mathcal{F}(v) + 0 + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx \geq \mathcal{F}(u),\end{aligned}$$

with strict inequality if $u \neq 0$. Thus, v is unique.

□

Now we prove that any minimizer is actually continuous.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, $\phi \in C^\infty(\Omega)$, and $v \in H^1(\Omega)$ be any minimizer of (2.1) subject to the boundary conditions $v|_{\partial\Omega} = g$. Then, $-\Delta v \geq 0$ in Ω.*

Proof. Let

$$\mathcal{F}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx.$$

Then, since v minimize $\mathcal{F}$ among all functions above the obstacle ϕ with fixed boundary conditions on $\partial\Omega$, we have that

$$\mathcal{F}(v+\varepsilon\eta) \geq \mathcal{F}(v) \quad \text{for every } \varepsilon \geq 0 \text{ and } \eta \geq 0,\ \eta \in C_c^\infty(\Omega).$$

This yields

$$\varepsilon\int_\Omega \nabla v\cdot\nabla\eta\,dx + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\eta|^2\,dx \geq 0 \quad \text{for every } \varepsilon \geq 0\text{and } \ \eta \geq 0,\ \eta \in C_c^\infty(\Omega)$$

and thus

$$\int_\Omega \nabla v\cdot\nabla\eta \geq 0 \quad \text{for every } \eta \geq 0,\ \eta \in C_c^\infty(\Omega).$$

This means that $-\Delta v \geq 0$ in Ω in the weak sense, as desired. □

From here, by showing first that $\{v > \phi\}$ is open, we obtain the Euler-Lagrange equations for the functional:

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, $\phi \in C^\infty(\Omega)$, and $v \in H^1(\Omega)$ be any minimizer of (2.1) subject to the boundary conditions $v|_{\partial\Omega} = g$. Then, $v \in C(\Omega)$ and satisfies*

$$\begin{cases} v & \geq & \phi \text{ in } \Omega \\ \Delta v & \leq & 0 \text{ in } \Omega \\ \Delta v & = & 0 \text{ in } \{v > \phi\} \cap \Omega, \end{cases} \tag{2.5}$$

Proof. By construction, we already know that $v \geq \phi$ in Ω and $-\Delta v \geq 0$ in Ω, i.e, v is weakly superharmonic. Up to replacing v in a set of measure zero, we may also assume that v is lower semi-continuous. Thusm we only need to prove that $\Delta v = 0$ in $\{v > \phi\} \cap \Omega$ and that v is, in fact, continuous.

In order to do that, first we prove that $\{v > \phi\} \cap \Omega$ is open. Let $x_0 \in \{v > \phi\} \cap \Omega$ be such that $v(x_0) - \phi(x_0) > \varepsilon_0 > 0$. By lower semi-continuity of v and by continuity of ϕ, there exists $\delta > 0$ such that $v(x) - \phi(x) > \varepsilon_0/2$ for all $x \in B_\delta(x_0)$, hence $B_\delta(x_0) \subset \{v > \phi\}$. Hence $\{v > \phi\} \cap \Omega$ is open since x_0 was arbitrary. This implies also that $\Delta v = 0$ in $\{v > \phi\} \cap \Omega$. Indeed, for any $x_0 \in \{v > \phi\}$ and $\eta \in C_c^\infty(B_\delta(x_0))$ with $|\eta| \leq 1$, we have $v \pm \varepsilon\eta \geq \phi$ in Ω for all $|\varepsilon| < \varepsilon_0/2$, and therefore it is an admissible competitor to the minimization problem. Thus, we have $\mathcal{F}(v + \varepsilon\eta) \geq \mathcal{F}(v)$ for all $|\varepsilon| < \varepsilon_0$, and differentiating in ε we deduce that v is harmonic in $\{v > \phi\} \cap \Omega$.

Finally we now show that v is continuous. By regularity of harmonic function we already know that v is continuous in $\{v > \phi\} \cap \Omega$. Let us show that v is continuous in $\{v = \phi\} \cap \Omega$. Let $y_0 \in \{v = \phi\} \cap \Omega$, and let us argue by contradiction. That is, since v is lower semi-continuous, let us assume that there is a sequence $y_k \to y_0$ such that $v(y_k) \to v(y_0) + \varepsilon_0 = \phi(y_0) + \varepsilon_0$ for some $\varepsilon_0 > 0$. Since ϕ is continuous, we may assume also that $y_k \in \{v > \phi\}$. Let us denote by z_k the projection of y_k towards $\{v = \phi\}$, so $\delta_k := |z_k - y_0| \leq 2|y_k - y_0| \downarrow 0$ and $v(z_k) \to v(y_0) = \phi(y_0)$. Now, since v is superharmonic it is true that

$$r \mapsto \fint_{B_r(x)} v(y)\, dy \quad \text{is monotone non-increasing for } r \in (0, \operatorname{dist}(x, \partial\Omega)),$$

thus using this fact

$$v(z_k) \geq \fint_{B_{2\delta_k}(y_k)} v = (1 - 2^{-n}) \fint_{B_{2\delta_k}(y_k) \setminus B_{\delta_k}(y_k)} v + 2^{-n} \fint_{B_{\delta_k}(y_k)} v = I_1 + I_2.$$

Observe that, for the first term, since v is lower semi-continuous and $\delta_k \downarrow 0$, we can assume that, for k large enough, $v \geq \phi(y_0) - 2^{-n}\varepsilon_0$ in $B_{2\delta_k}$, so that $I_1 \geq (1 - 2^{-n})[\phi(y_0) - 2^{-n}\varepsilon_0]$. On the other hand, since v is harmonic in $B_{\delta_k}(y_k)$, we have by mean value property that $I_2 = 2^{-n}v(y_k)$. Combining everything, we get

$$v(z_k) \geq (1 - 2^{-n})[\phi(y_0) - 2^{-n}\varepsilon_0] + 2^{-n}v(y_k) \to \phi(y_0) + 2^{-2n}\varepsilon_0$$

which contradicts the fact that we had $v(z_k) \to v(y_0) = \phi(y_0)$. Hence, v is continuous in Ω.

□

Optimal regularity of solutions. From now on, we will actually localize the problem and study it in a ball:

$$\begin{cases} v & \geq & \phi \text{ in } B_1 \\ \Delta v & \leq & 0 \text{ in } B_1 \\ \Delta v & = & 0 \text{ in } \{v > \phi\} \cap B_1. \end{cases} \tag{2.6}$$

We want to answer the following question:

What is the optimal regularity of solutions?

Notice that in the set $\{v > \phi\}$ the solution is harmonic, i.e. $\Delta v = 0$, while in the interior of $\{v = \phi\}$ we have $\Delta v = \Delta \phi$. Thus, since $\Delta\phi$ is in general not zero, Δv is *discontinuous* across the free boundary $\partial\{v > \phi\}$ in general. In particular, $v \notin C^2$.

We will prove that any minimizer of (2.1) is actually $C^{1,1}$, which gives the answer to the previous question.

Theorem 2.4 (Optimal regularity)**.** *Let $\phi \in C^\infty(B_1)$, and v be any solution to (2.6). Then $v \in C^{1,1}$ in $B_{1/2}$, with the estimate*

$$\|v\|_{C^{1,1}(B_{1/2})} \leq C\big(\|v\|_{L^\infty(B_1)} + \|\phi\|_{C^{1,1}(B_1)}\big).$$

The constant C depends only on n.

To prove this we need the following lemma.

Lemma 2.5. *Let $\phi \in C^\infty(B_1)$, and v be any solution to (2.6). Let $x_0 \in \overline{B_{1/2}}$ be any point on $\{v = \phi\}$. Then, for any $r \in (0, \frac{1}{4})$ we have*

$$0 \leq \sup_{B_r(x_0)} \big(v - \phi\big) \leq Cr^2,$$

with C depending only on n and $\|\phi\|_{C^{1,1}(B_1)}$.

Proof. Without loss of generality we can assume $\|\phi\|_{C^{1,1}(B_1)} \leq 1$.

Let $l(x) := \phi(x_0) + \nabla\phi(x_0) \cdot (x - x_0)$ be the linear part of ϕ at x_0. Let $r \in (0, \frac{1}{4})$. Then by $C^{1,1}$ regularity of ϕ, in $B_r(x_0)$ we have

$$l(x) - r^2 \leq \phi(x) \leq v(x).$$

We want to show that, in the bakk $B_r(x_0)$ we have

$$v(x) \leq l(x) + Cr^2$$

For this, consider

$$w(x) := v(x) - [l(x) - r^2].$$

This function w satisfies $w \geq 0$ in $B_r(x_0)$, and $-\Delta w = -\Delta v \geq 0$ in $B_r(x_0)$.

Let us split w into

$$w = w_1 + w_2,$$

with

$$\begin{cases} \Delta w_1 & = 0 \text{ in } B_r(x_0) \\ \quad w_1 & = w \text{ on } \partial B_r(x_0) \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_2 & \geq 0 \text{ in } B_r(x_0) \\ \quad w_2 & = 0 \text{ on } \partial B_r(x_0). \end{cases}$$

Notice that

$$0 \leq w_1 \leq w \text{ and } 0 \leq w_2 \leq w.$$

We have that

$$w_1(x_0) \leq w(x_0) = v(x_0) - [l(x_0) - r^2] = r^2,$$

and by the Harnack inequality we get

$$\|w_1\|_{L^\infty(B_{r/2}(x_0)} \leq Cr^2.$$

For w_2, notice that $\Delta w_2 = \Delta v$, and in particular $\Delta w_2 = 0$ in $\{v > \phi\}$. This means that w_2 attains its maximum on $\{v = \phi\}$. But in the set $\{v = \phi\}$ we have

$$w_2 \leq w = \phi - [l - r^2] \leq Cr^2.$$

and therefore we deduce that

$$\|w_2\|_{L^\infty(B_r(x_0))} \leq Cr^2.$$

Combining the bounds for w_1 and w_2, we get $\|w\|_{L^\infty(B_r(x_0))} \leq Cr^2$. Translating this into v, and using that $\|\phi\|_{C^{1,1}(B_1)} \leq 1$, we find $v - \phi \leq Cr^2$ in $B_{r/2}(x_0)$. □

Therefore, we proved that:

At every free boundary point x_0, v separetes from ϕ at most quadratically.

We will see that this implies the $C^{1,1}$ regularity.

Proof of Theorem 2.4. Dividing v by a constant if necessary, we may assume that $\|v\|_{L^\infty(B_1)} + \|\phi\|_{C^{1,1}(B_1)} \leq 1$. We already know that $v \in C^\infty$ in the set $\{v > \phi\}$ (since v is harmonic), and also in the interior of the set $\{v = \phi\}$, (since ϕ is C^∞). Moreover, on the interface $\Gamma = \partial\{v > \phi\}$ we have proved the quadratic growth $\sup_{B_r(x_0)}(v - \phi) \leq Cr^2$. Let us prove that this yields the $C^{1,1}$ bound we want. Let $x_1 \in \{v > \phi\} \cap B_{1/2}$, and let $x_0 \in \Gamma$ be the closest free boundary point. Denote $\rho = |x_1 - x_0|$. Then, we have $\Delta v = 0$ in $B_\rho(x_1)$(mettere figura), and thus we have also $\Delta(v-l)=0$, where l is the linear part of ϕ at x_0.

By estimates for harmonic functions, we find

$$\|D^2 v\|_{L^\infty(B_{\rho/2}(x_1))} = \|D^2(v-l)\|_{L^\infty/(B_{\rho/2}()x_1)} \le \frac{C}{\rho^2}\|v-l\|_{L^\infty/(B_\rho()x_1)}.$$

But by the growth proved in the previous Lemma, we have $\|v-l\|_{L^\infty(B_\rho(x_1))} \le C\rho^2$, which yields

$$\|D^2 v\|_{L^\infty(B_{\rho/2}(x_1))} \le \frac{C}{\rho^2}\rho^2 = C.$$

In particular, $\|D^2 v(x_1)\| \le C$. We can do this for all $x_1 \in \{v > \phi\} \cap B_{1/2}$, and on $\partial\{v > \phi\}$ we have quadratic growth by Lemma 2.5, hence it follows that $\|v\|_{C^{1,1}(B_{1/2})} \le C$, as wanted. □

Nondegeneracy. We now want to prove that, at all free boundaries points, v separates from ϕ at least quadratically.

That is, we want

$$0 < cr^2 \le \sup_{B_r(x_0)} \big(v-\phi\big) \le Cr^2, \tag{2.7}$$

for all free boundary points $x_0 \in \partial\{v > \phi\}$.

Remark 2.6. *Since $-\Delta v \ge 0$ everywhere, its is clear that if $x_0 \in \partial\{v > \phi\}$ is a free boundary point, then necessarily $-\Delta\phi(x_0) \ge 0$, since v touches ϕ from above at x_0.*

Proposition 2.7 (Nondegeneracy). *Let $\phi \in C^\infty(B_1)$, and v be any solution to (2.6). Assume that ϕ satisfies $-\Delta\phi \ge c_0 > 0$ in B_1. Then, for every free boundary point $X_0 \in \{v > \phi\} \cap B_{1/2}$, we have*

$$0 < cr^2 \le \sup_{B_r(x_0)} (v-\phi) \le Cr^2 \quad \textit{for all } r \in (0, \tfrac{1}{4}),$$

with a constant $c > 0$ depending only on n and c_0.

Proof. Let $x_1 \in \{v > \phi\}$ be any point close to x_0 (we will then let $x_1 \to x_0$ at the end of the proof). Consider the funtion

$$w(x) := v(x) - \phi(x) - \frac{c_0}{2n}|x-x_1|^2$$

Then, in $\{v > \phi\}$ we have

$$\Delta w = \Delta v - \Delta\phi - c_0 = -\Delta\phi - c_0 \ge 0$$

and hence $-\Delta w \le 0$ in $\{v > \phi\} \cap B_r(x_1)$. Moreover, $w(x_1) > 0$.

By the maximum principle, w attains a positive maximum on $\partial(\{v > \phi\} \cap B_r(x_1))$. But on the free boundary $\partial\{v > \phi\}$ we clearly have $w < 0$. Therefore, there is a point on $\partial B_r(x_1)$ at which $w > 0$. In other words,

$$0 < \sup_{\partial B_r(x_1)} w = \sup_{\partial B_r(x_1)} (v - \phi) - \frac{c_0}{2n} r^2$$

□

Summary of basic properties. Let v be any solution to the obstacle problem

$$\begin{cases} v & \geq & \phi \text{ in } B_1 \\ \Delta v & \leq & 0 \text{ in } B_1 \\ \Delta v & = & 0 \text{ in } \{v > \phi\} \cap B_1. \end{cases}$$

Then, we have:

- Optimal regularity: $\|v\|_{C^{1,1}(B_{1/2})} \big(\|v\|_{L^\infty(B_1)} + \|\phi\|_{C^{1,1}(B_1)}\big)$
- Nondegeneracy: if $-\Delta\phi \geq c_0 > 0$, then

  $$0 < cr^2 \geq \sup_{B_r(x_0)} (v - \phi) \leq Cr^2 \quad \text{for all } r \in (0, \frac{1}{2})$$

  at all free boundary points $x_0 \in \partial\{v > \phi\} \cap B_{1/2}$.
- Equivalence with zero obstacle: The problem is equivalent to

  $$\begin{cases} u & \geq & 0 \text{ in } B_1 \\ \Delta u & \leq & f \text{ in } B_1 \\ \Delta u & = & f \text{ in } \{u > \ 0\} \cap B_1. \end{cases}$$

  where $f = -\Delta\phi \geq c_0 > 0$.

We will next provide an alternative approach to the optimal regularity.

2.2 Basic properties of Solutions II

We proceed now to study the basic properties of solutions $u \geq 0$ to the obstacle problem (2.4).

Throughout this section we will always assume

$$f \geq 0 \quad \text{in } \Omega.$$

We can prove the existence of solutions with the same method used in the previous section.

Proposition 2.8 (Existence and uniqueness)**.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ be such that*

$$\mathcal{C} = \left\{ u \in H^1(\Omega) \;\; u \geq 0 \text{ in } \Omega,\; u|_{\partial\Omega} = g \right\} \neq \varnothing$$

Then, for any $f \in L^2(\Omega)$ there exists a unique minimizer of

$$\frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega fu\, dx$$

among all functions $u \in H^1(\Omega)$ satisfying $u \geq 0$ in Ω and $u|_{\partial\Omega} = g$.

Proof. Let

$$I := \inf\left\{ \frac{1}{2}\int_\Omega |\nabla w|^2\, dx + \int_\Omega fw \;:\; w \in H^1(\Omega),\; w|_{\partial\Omega} = g,\; w \geq 0 \text{ in } \Omega \right\},$$

that is, the infimum value of $\mathcal{F}(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\, dx + \int_\Omega fw$ among all admissible functions w. Notice that, by Hölder's inequality, $\mathcal{F}(w) < +\infty$ if $w \in H^1(\Omega)$.
We take again a sequence of functions $\{v_k\}$ such that $v_k \in H^1(\Omega)$, $v_k|_{\partial\Omega} = g,\; v_k \geq 0$ in Ω, and $\mathcal{F}(v_k) \to I$ as $k \to \infty$. By Poincaré inequality, Hölder's inequality, and the fact that $\mathcal{F}(v_k) \leq I + 1$, for k large enough

$$\begin{aligned} \|v_k\|^2_{H^1(\Omega)} &\leq C\left(\int_\Omega |\nabla v_k|^2 + \int_{\partial\Omega} g^2 \right) \leq C\left(I + 1 + \int_\Omega |fv_k| + \frac{1}{2}\int_{\partial\Omega} g^2 \right) \\ &\leq C\left(I + 1 + \|f\|_{L^2(\Omega)}\|v_k\|_{H^1(\Omega)} + \frac{1}{2}\int_{\partial\Omega} g^2 \right). \end{aligned}$$

In particular, $\|v_k\|_{H^1(\Omega)} \leq C$ for some constant C depending only on n, Ω, g, f, and I. Hence, a subsequence v_{k_j} converges to a certain function v strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. By compactness of the trace operator $v_{k_j}|_{\partial\Omega} \to v|_{\partial\Omega} = g$ in $L^2(\Omega)$. Furthermore, v satisfies $\mathcal{F}(v) \leq \liminf_{j\to\infty}(v_{k_j})$, and therefore it will be a minimizer of the energy functional. Since $v_{k_j} \geq 0$ in Ω and $v_{k_j} \to v$ in $L^2(\Omega)$, we have $v \geq 0$ in Ω. Thus, there is a minimizer v.
The uniqueness of the minimizer follows frome the stric convexity of the functional $\mathcal{F}$. □

Furthermore, we have the following equivalence. (Recall that we denote $u^+ = \max\{u, 0\}$, and $u^- = \max\{-u, 0\}$, so that $u = u^+ - u^-$).

Proposition 2.9. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ be such that*

$$\mathcal{C} = \left\{ u \in H^1(\Omega) : u \geq 0 \text{ in } \Omega,\; u|_{\partial\Omega} = g \right\} \neq \varnothing.$$

Then, the following are equivalent.

(i) u minimizes $\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu$ among all functions satisfying $u \geq 0$ in Ω and $u|_{\partial\Omega} = g$.

(ii) u minimizes $\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+$ among all functions satisfying $u|_{\partial\Omega} = g$.

Proof. The two functionals coincide whenever $u \geq 0$. Thus, the only key point is to prove that the minimizer in (ii) must be nonnegative, i.e., $u = u^+$. (Notice that $\mathcal{C} \neq \varnothing$ implies that $g \geq 0$ on Ω.) To show this, recall that the positive part of any H^1 function is still in H^1, and moreover $|\nabla u|^2 = |\nabla u^+|^2 + |\nabla u^-|^2$. Thus, we have that (recall that $f \geq 0$ in Ω)

$$\frac{1}{2}\int_\Omega |\nabla u^+|^2 + \int_\Omega fu^+ \leq \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+,$$

with strict inequality unless $u = u^+$. This means that any minimizer u of the functional in (ii) must be nonnegative, and thus we are done. □

Let us next prove that any minimizer of (2.4) is actually a solution to (mettere equazione) below.

We recall that we are always assuming that obstacles are as smooth as necessary, $\varphi \in C^\infty(\Omega)$, and therefore we assume here that $f \in C^\infty(\Omega)$ as well.

Proposition 2.10. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, $f \in C^\infty(\Omega)$, and $u \in H^1(\Omega)$ be any minimizer of (2.4) subject to the boundary conditions $u|_{\partial\Omega} = g$.*

Then, u solves

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in} \quad \Omega \tag{2.8}$$

in the weak sense.

Proof. Notice that, by Proposition 2.9, u is actually a minimizer of

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+$$

subject to the boundary conditions $u|_{\partial\Omega} = g$.

Thus, for any $\eta \in H^1_0(\Omega)$ and $\varepsilon > 0$ we have

$$\mathcal{F}(u + \varepsilon\eta) \geq \mathcal{F}(u).$$

In particular, we obtain

$$0 \leq \lim_{\varepsilon\downarrow 0} \frac{\mathcal{F}(u+\varepsilon\eta) - \mathcal{F}(u)}{\varepsilon} = \int_\Omega \nabla u \cdot \nabla \eta + \lim_{\varepsilon\downarrow 0}\int_\Omega f\frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon}.$$

Notice that

$$\lim_{\varepsilon\downarrow 0} \frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon} = \begin{cases} \eta & \text{in} \quad \{u > 0\} \\ \eta^+ & \text{in} \quad \{u = 0\}, \end{cases}$$

so that we have

$$\int_\Omega \nabla u\cdot\nabla\eta + \int_\Omega f\eta\chi_{\{u>0\}} + \int_\Omega f\eta^+\chi_{\{u=0\}} \geq 0 \quad \text{for all} \quad \eta\in H_0^1(\Omega).$$

Assume first that $\eta\geq 0$, so that

$$\int_\Omega \nabla u\cdot\nabla\eta + \int_\Omega f\eta \geq 0 \quad \text{for all} \quad \eta\in H_0^1(\Omega), \quad \eta\geq 0,$$

which implies that $\Delta u\leq f$ in the weak sense. On the other hand, if $\eta\leq 0$, then

$$\int_\Omega \nabla u\cdot\nabla\eta + \int_\Omega f\eta\chi_{\{u>0\}} \geq 0 \quad \text{for all} \quad \eta\in H_0^1(\Omega), \quad \eta\leq 0,$$

which implies that $\Delta u\geq f\chi_{\{u>0\}}$ in the weak sense. In all (recall that $f\geq 0$),

$$f\chi_{\{u>0\}}\leq \Delta u\leq f \quad \text{in} \quad \Omega.$$

(In particular, notice that $\Delta u = f$ in $\{u>0\}$.) Now, since f is smooth, this implies that $\Delta u\in L^\infty_{\rm loc}(\Omega)$. By Proposition ?? we deduce that $u\in C^{1,1-\varepsilon}$ for every $\varepsilon>0$. Moreover, since $\Delta u\in L^\infty_{\rm loc}(\Omega)$ we have $\Delta u\in L^2_{\rm loc}(\Omega)$ and by Calderón-Zygmund estimates we have $u\in W^{2,2}_{\rm loc}(\Omega)$. Thus, $\Delta u=0$ almost everywhere in the level set $\{u=0\}$ and we have

$$\Delta u = f\chi_{\{u>0\}} \quad \text{a.e. in} \quad \Omega.$$

From here we deduce that $\Delta u = f\chi_{\{u>0\}}$ in Ω in the weak sense.

□

Notice that in the previous Section, when dealing with minimizers v of (2.1), it was not easy to prove that v is continuous. Here, instead, thanks to Proposition 2.4 we simply used Schauder-type estimates for the Laplacian to directly deduce that u is $C^{1,1-\varepsilon}$, which is the almost-optimal regularity of solutions.

Alternatively we could prove the regularity in a different way as shown below. For the sake of simlicity we assume $f=1$.

Let $D_0\subset\mathbb{R}^n$ be a bounded open set with smooth boundary. Note that, by a standard concentration-compactness, there is a solution $V\in H^1(\mathbb{R}^n)$ of the auxilary problem

$$\min\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla v|^2+v\right)dx \;:\; v\in H^1(\mathbb{R}^n)\,, v=1 \text{ on } D_0,\; v\geq 0 \text{ on } \mathbb{R}^n\right\} \tag{2.9}$$

We are going to prove that the set $\Omega=\{V>0\}$ is open, and that V is continuous.

Remark 2.11 (Truncation). We note that any solution V to (2.9) automatically satisfies $V\leq 1$.

Remark 2.12 (Radial solutions). Suppose that $D_0 \subset \mathbb{R}^n$ is a ball of radius r_0. Then the minimizer V is radially symmetric and has compact support. In fact the radial symmetry $V(x) = V(|x|) = V(r)$ follows by a Schwartz symmetrization. Thus V satisfies

$$\begin{cases} V'' + \frac{n-1}{r}V' & = 1 \quad \text{on} \quad (r_0, \infty) \cap \{V > 0\}, \\ 0 \leq V \leq 1, V(r_0) & = 1, \quad V' \leq 0. \end{cases}$$

Multiplying both sides by V' we get

$$\frac{1}{2}\Big[|V'(r)|^2\Big]' + \frac{n-1}{r}|V'(r)|^2 = -|V'(r)|,$$

and so taking $f(r) = |V'(r)|^2$, we get

$$f'(r) \leq -2\sqrt{f(r)} \quad \text{on} \quad (r_0, \infty) \cap \{V > 0\},$$

and so

$$|V'(r)| = \sqrt{f(r)} \leq C - r,$$

for some constant $C > 0$, which gives the compactness of the support V.

Remark 2.13 (Comparison). Suppose that $\Omega_0 \subset \Omega_1$ are two given measurable sets and that the functions V_i, for $i = 0, 1$ are minimizers respectively of

$$\min \left\{ \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla v|^2 + v\right) dx \ : \ v \in H^1(\mathbb{R}^n)\ , v = 1 \text{ on } \Omega_i,\ v \geq 0 \text{ on } \mathbb{R}^n \right\}. \quad (2.10)$$

Then $V_1 \geq V_0$.

Remark 2.14 (Compact support). Suppose that V_0 is a solution of (2.9) for a given bounded measurable $D_0 \subset \mathbb{R}^n$. Then, V_0 has compact support.

Remark 2.15 (Subharmonicity). . Suppose that V is a minimizer of (2.9). Then, V is subharmonic on the open set $\mathbb{R}^n \setminus \overline{D_0}$. Let $u \in H^1(\mathbb{R}^n)$ be such that $V \geq u$ and $u - V \in H^1_0(\mathbb{R}^n \setminus \overline{D_0})$. Then

$$\int_{\mathbb{R}^n} |\nabla u|^2\, dx \geq \int_{\mathbb{R}^n} |\nabla u^+|^2\, dx \geq \int_{\mathbb{R}^n} |\nabla V|^2 + 2(V - u^+)\, dx \geq \int_{\mathbb{R}^n} |\nabla V|^2\, dx.$$

Remark 2.16 (Superharmonicity). Suppose that V is a minimizer of (2.9). Then $\Delta V \leq 1$ on $\mathbb{R}^n$. Let $\varphi \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ be non-negative. Then for every $\varepsilon > 0$

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla(V + \varepsilon\varphi)|^2 + V + \varepsilon\varphi\right) dx \geq \int_{\mathbb{R}^n} \frac{1}{2|\nabla V|^2 + V}\, dx,$$

which gives

$$\int_{\mathbb{R}^n} \nabla V \cdot \nabla\varphi\, dx + \int_{\mathbb{R}^n} \varphi\, dx \geq 0.$$

Remark 2.17 (Continuity). Since $V \in H^1(\mathbb{R}^n)$ satisfies

$$0 \leq \Delta V \leq 1, \quad \text{on} \quad \mathbb{R}^n \setminus \overline{D_0},$$

it is continuous on $\mathbb{R}^n \setminus \overline{D_0}$.

Remark 2.18 (Diameter). Since $\Delta V \leq 1$, for every $x_0 \in \mathbb{R}^n$ such that $\mathrm{dist}(x_0, D_0) \geq \sqrt{2n}$, we have that

$$V(x) \leq \frac{|x - x_0|^2}{2n},$$

and so $V = 0$ on the set $\{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, D_0) > \sqrt{2n}\}$.

Remark 2.19 (Inner ball condition). If we suppose that D_0 satisfies the inner ball condition for a radius r_0, then, by the comparison principle, we have that $\mathrm{dist} D_0, \{V = 0\} \geq c r_0$ for some dimensional constant c.

Remark 2.20 (Behaviour of V near the free boundary $\partial\{V > 0\}$). Let $x_0 \in \partial\{V > 0\}$, $0 < r < \mathrm{dist}(D_0, \{V = 0\})$ and $h_V \in B_r(x_0)$ be the solution of

$$\Delta h_v = 0 \quad \text{in} \quad B_r(x_0), \quad h_V = V \quad \text{on} \quad \partial B_r(x_0).$$

Then, we have $h_V \geq V$ and $-\Delta(h_V - V) \leq 1$ in $B_r(x_0)$, which gives that

$$\fint_{B_r(x_0)} h_V \, dx = h_V(x_0) \leq \frac{r^2}{2n}.$$

For each $y \in B_{r/2}(x_0)$ we have

$$V(y) \leq h_V(y) = \fint_{B_{r/2}(y)} h_V \, dx \leq 2^n \fint_{B_r(x_0)} h_V \, dx \leq \frac{2^{n-1} r^2}{n}.$$

Remark 2.21 (Behaviour of ∇V near the free boundary $\partial\{V > 0\}$). . Consider the function $\phi : [0, 1] \to \mathbb{R}$ defined as

$$\phi(r) = \begin{cases} \frac{1}{4} - \frac{r^2}{2}, \text{ for } & r \in [0, 1/2] \\ \frac{(r-1)^2}{2}, \text{ for } & r \in [1/2, 1]. \end{cases}$$

We note that the funciton $\Phi(x) := \phi(|x|)$ satisfies $\nabla \Phi = 0$ on ∂B_1 and

$$\Delta \Phi(x) = \partial_{rr}\phi + \frac{n-1}{r}\partial_r \phi = -n\chi_{B_{1/2}} + \Big(n - \frac{n-1}{r}\chi_{B_1 \setminus B_{1/2}}\Big).$$

Let $x_0 \in \partial\{V > 0\}$. Without loss of generality we can suppos $x_0 = 0$. Let $\Phi_r(x) := r^2 \Phi(\frac{x}{r})$ and consider the test function $V\Phi_r \in H^1_0(\{V > 0\} \cap B_r)$.

$$\begin{aligned} -\int_{B_r} V\Phi_r \, dx &= \int_{B_r} \nabla V \cdot \nabla(V\Phi_r) \, dx \\ &= \int_{B_r} |\nabla V|^2 \Phi_r \, dx + \frac{1}{2}\int_{B_r} \nabla(V^2) \cdot \nabla \Phi_r \, dx \\ &= \int_{B_r} |\nabla V|^2 \Phi_r \, dx - \frac{1}{2}\int_{B_r} V^2 \Delta \Phi_r \, dx. \end{aligned}$$

Thus, we have

$$\frac{r^2}{2^{n+3}}\fint_{B_{r/2}} |\nabla V|^2\, dx \le \frac{n}{2}\fint_{B_r} V^2\, dx \le \frac{2^{2n+3}}{n} r^4,$$

and so

$$\fint_{B_{r/2}} |\nabla V|^2\, dx \le \frac{2^{3n+6}}{n} r^2.$$

Remark 2.22 ($V \in C^1(\mathbb{R}^n \setminus \overline{D_0})$). Each component $V_i = \frac{\partial V}{\partial x_i}$ of the gradient ∇V is an harmonic function in $\{V > 0\} \setminus \overline{D_0}$. Moreover, from the last inequality we have $V_i(x) \to 0$ as $\operatorname{dist}(x, \{V = 0\}) \to 0$, which gives that V_i is continuous on $\mathbb{R}^n \setminus \overline{D_0}$.

Remark 2.23 (Nondegeneracy of V). Let $x_0 \in \{V > 0\} \setminus \overline{D_0}$. The function

$$U(x) = V(x) - \frac{|x - x_0|^2}{2n},$$

is harmonic in $\{V > 0\} \setminus \overline{D_0}$. Then, by the maximum principle

$$V(x_0) \le \sup_{x \in \{V>0\} \cap \partial B_r(x_0)} V(x) - \frac{r^2}{2n}.$$

Since the same estimate holds for every $x_0 \in \{V > 0\} \setminus \overline{D_0}$, we get that it holds also for $x_0 \in \partial\{V > 0\}$.

Optimal regularity of solutions

Thanks to the previous results, we know that any minimizer is continuous and solves (2.8).

From now on, we will localize the problem and study it in a ball:

$$\begin{cases} u \ \ge \ 0 & \text{in } B_1 \\ \Delta u \ = \ f\chi_{\{u>0\}} & \text{in } B_1. \end{cases} \tag{2.11}$$

Our next goal is to answer the following question:

Question: *What is the optimal regularity of solutions?*

First, a few important considerations. Notice that in the set $\{u > 0\}$ we have $\Delta u = f$, while in the interior of $\{u = 0\}$ we have $\Delta u = 0$ (since $u \equiv 0$ there).

Thus, since f is in general not zero, Δu is *discontinuous* across the free boundary $\partial\{u > 0\}$ in general. In particular, $u \notin C^2$.

We will now prove that any minimizer of (2.4) is actually $C^{1,1}$, which gives the:

Answer: $u \in C^{1,1}$ *(second derivatives are bounded but not continuous)*

The precise statement and proof are given next.

Theorem 2.24 (Optimal regularity). *Let $f \in C^\infty(B_1)$, and let u be any solution to* (2.11)*. Then, u is $C^{1,1}$ inside $B_{1/2}$, with the estimate*

$$\|u\|_{C^{1,1}(B_{1/2})} \le C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{\mathrm{Lip}(B_1)}\big).$$

The constant C depends only on n.

To prove this, the main step is the following.

Lemma 2.25. *Let u be any solution to* (2.11)*. Let $x_\circ \in \overline{B_{1/2}}$ be any point on $\{u = 0\}$. Then, for any $r \in (0, \frac{1}{4})$ we have*

$$0 \le \sup_{B_r(x_\circ)} u \le Cr^2,$$

with C depending only on n and $\|f\|_{L^\infty(B_1)}$.

Proof. We have that $\Delta u = f\chi_{\{u>0\}}$ in B_1, with $f\chi_{\{u>0\}} \in L^\infty(B_1)$. Thus, since $u \ge 0$, we can use the Harnack inequality for the equation $\Delta u = f\chi_{\{u>0\}}$ in $B_{2r}(x_\circ)$, to find

$$\sup_{B_r(x_\circ)} u \le C\left(\inf_{B_r(x_\circ)} u + r^2\|f\chi_{\{u>0\}}\|_{L^\infty(B_{2r}(x_\circ))}\right).$$

Since $u \ge 0$ and $u(x_\circ) = 0$, this yields $\sup_{B_r(x_\circ)} u \le C\|f\|_{L^\infty(B_1)}r^2$, as wanted. □

We have proved the following:

At every free boundary point $x_\circ$, u grows (at most) quadratically.

We will see that this implies the $C^{1,1}$ regularity.

Proof of Theorem 2.24. Dividing u by a constant if necessary, we may assume that $\|u\|_{L^\infty(B_1)} + \|f\|_{\mathrm{Lip}(B_1)} \le 1$.

We already know that $u \in C^\infty$ in the set $\{u > 0\}$ (since $\Delta u = f \in C^\infty$ there), and also inside the set $\{u = 0\}$ (since $u = 0$ there). Moreover, on the interface $\Gamma = \partial\{u > 0\}$ we have proved the quadratic growth $\sup_{B_r(x_\circ)} u \le Cr^2$. Let us prove that this yields the $C^{1,1}$ bound we want.

Let $x_1 \in \{u > 0\} \cap B_{1/2}$, and let $x_\circ \in \Gamma$ be the closest free boundary point. Denote $\rho = |x_1 - x_\circ|$. Then, we have $\Delta u = f$ in $B_\rho(x_1)$.

By Schauder estimates, we find

$$\|D^2u\|_{L^\infty(B_{\rho/2}(x_1))} \le C\left(\frac{1}{\rho^2}\|u\|_{L^\infty(B_\rho(x_1))} + \|f\|_{\mathrm{Lip}(B_1)}\right).$$

But by the growth proved in the previous Lemma, we have $\|u\|_{L^\infty(B_\rho(x_1))} \le C\rho^2$, which yields

$$\|D^2u\|_{L^\infty(B_{\rho/2}(x_1))} \le C.$$

In particular,

$$|D^2u(x_1)| \le C.$$

We can do this for each $x_1 \in \{u > 0\} \cap B_{1/2}$, and therefore $\|u\|_{C^{1,1}(B_{1/2})} \le C$, as wanted. □

Also, notice that as a consequence of the previous results, we have that as soon as the solution to (2.11) has non-empty contact set, then its $C^{1,1}$ norm is universally bounded.

Corollary 2.26. *Let u be any solution to* (2.11)*, and let us assume that* $u(0) = 0$ *and* $\|f\|_{\mathrm{Lip}(B_1)} \le 1$*. Then,*

$$\|u\|_{C^{1,1}(B_{1/2})} \le C$$

for some C depending only on n.

Proof. It is an immediate consequence of Theorem 2.24 combined with Lemma 2.25. □

Nondegeneracy. For completeness, we now state the nondegeneracy in this setting. That is, at all free boundary points, u grows *at least* quadratically (we already know *at most* quadratically). We want:

$$0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2$$

for all free boundary points $x_\circ \in \partial\{u > 0\}$.

This property is essential in order to study the free boundary later. As before, for this we need the following natural assumption:

Assumption: *The right-hand side f satisfies*

$$f \ge c_\circ > 0$$

in the ball B_1.

Proposition 2.27 (Nondegeneracy)**.** *Let u be any solution to* (2.11)*. Assume that $f \ge c_\circ > 0$ in B_1. Then, for every free boundary point $x_\circ \in \partial\{u > 0\} \cap B_{1/2}$, we have*

$$0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2 \qquad \text{for all } r \in (0, \tfrac{1}{2}),$$

with a constant $c > 0$ depending only on n and $c_\circ$.

Summary of basic properties. Let u be any solution to the obstacle problem

$$\begin{cases} u \ge 0 & \text{in } B_1, \\ \Delta u = f\chi_{\{u>0\}} & \text{in } B_1. \end{cases}$$

Then, we have:

- Optimal regularity: $\|u\|_{C^{1,1}(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{\mathrm{Lip}(B_1)}\right)$
- Nondegeneracy: If $f \ge c_\circ > 0$, then

$$0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2 \qquad \text{for all } r \in (0, \tfrac{1}{2})$$

at all free boundary points $x_\circ \in \partial\{u > 0\} \cap B_{1/2}$.

Using these properties, we can now start the study of the free boundary.

Chapter 3

Regularity of free boundary

3.1 Regularity of free boundaries: an overview

From now on, we consider any solution to

$$\begin{cases} u \in C^{1,1}(B_1), \\ u \geq 0 \quad \text{in } B_1, \\ \Delta u = f \quad \text{in } \{u > 0\}, \end{cases} \tag{3.1}$$

(see Figure 3.1) with

$$f \geq c_\circ > 0 \qquad \text{and} \qquad f \in C^\infty. \tag{3.2}$$

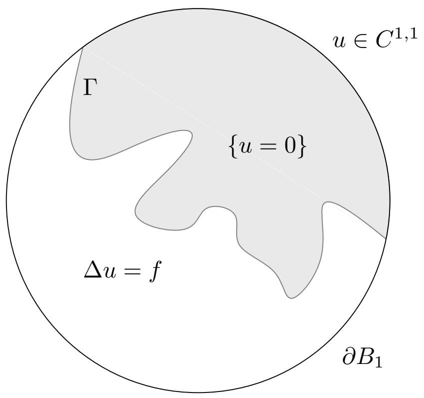

Figure 3.1: A solution to the obstacle problem in B_1.

Notice that on the interface

$$\Gamma = \partial\{u > 0\} \cap B_1$$

we have that

$$\begin{aligned} u &= 0 \quad \text{on } \Gamma, \\ \nabla u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

The central mathematical challenge in the obstacle problem is to

understand the geometry/regularity of the free boundary Γ.

Notice that, even if we already know the optimal regularity of u (it is $C^{1,1}$), we know nothing about the free boundary Γ. A priori Γ could be a very irregular object, even a fractal set with infinite perimeter.

As we will see, under the natural assumption $f \geq c_\circ > 0$, it turns out that free boundaries are always smooth, possibly outside a certain set of singular points. In fact, in our proofs we will assume for simplicity that $f \equiv 1$ (or constant). We do that in order to avoid x-dependence and the technicalities associated to it, which gives cleaner proofs. In this way, the main ideas behind the regularity of free boundaries are exposed.

Main results: Assume from now on that u solves (3.1)-(3.2). Then, the main known results on the free boundary $\Gamma = \partial\{u > 0\}$ can be summarized as follows:

• At every free boundary point $x_\circ \in \Gamma$, we have

$$0 < cr^2 \leq \sup_{B_r(x_\circ)} u \leq Cr^2 \qquad \forall r \in (0, r_\circ)\,.$$

• The free boundary Γ splits into *regular points* and *singular points.*

• The set of *regular points* is an open subset of the free boundary, and Γ is C^∞ near these points.

• *Singular points* are those at which the contact set $\{u = 0\}$ has *zero density*, and these points (if any) are contained in an $(n-1)$-dimensional C^1 manifold.

Summarizing, *the free boundary is smooth, possibly outside a certain set of singular points.* See Figure 3.2.

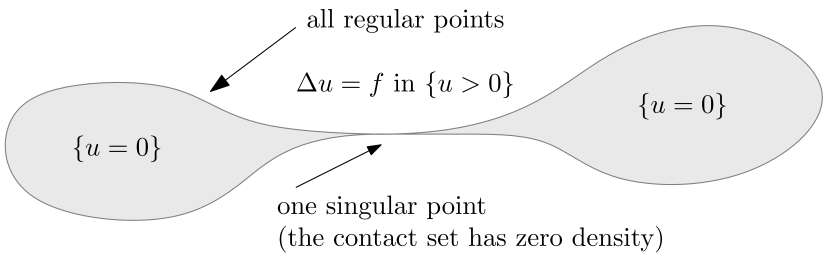

Figure 3.2: Singular points are those where the contact set has zero density.

So far, we have not even proved that Γ has finite perimeter, or anything at all about Γ. Our goal will be to prove that Γ *is* C^∞ *near regular points.*

Overview of the strategy

To prove these regularity results for the free boundary, one considers *blow-ups.* Namely, given any free boundary point $x_\circ$ for a solution u of (3.1)-(3.2), one takes the rescalings

$$u_r(x) := \frac{u(x_\circ + rx)}{r^2},$$

with $r > 0$ small. This is like "zooming in" at a free boundary point.

The factor r^{-2} is chosen so that

$$\|u_r\|_{L^\infty(B_1)} \approx 1$$

as $r \to 0$; recall that $0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2$.

Then, by $C^{1,1}$ estimates, we will prove that a subsequence of u_r converges to a function u_0 locally uniformly in $\mathbb{R}^n$ as $r \to 0$. Such function u_0 is called a *blow-up of u at $x_\circ$*.

Any blow-up u_0 is a *global* solution to the obstacle problem, with $f \equiv 1$ (or with $f \equiv$ constant > 0).

Then, the main issue is to *classify blow-ups*: that is, to show that

$$\begin{aligned} \text{either} \qquad & u_0(x) = \tfrac{1}{2}(x \cdot e)_+^2 \qquad && \text{(this happens at regular points)} \\ \text{or} \qquad & u_0(x) = \tfrac{1}{2}x^T A x && \text{(this happens at singular points).} \end{aligned}$$

Here, $e \in \mathbb{S}^{n-1}$ is a unit vector, and $A \ge 0$ is a positive semi-definite matrix satisfying $\mathrm{tr}A = 1$. Notice that the contact set $\{u_0 = 0\}$ becomes a half-space in case of regular points, while it has zero measure in case of singular points;

Once this is done, one has to "transfer" the information from the blow-up u_0 to the original solution u. Namely, one shows that, in fact, the free boundary is $C^{1,\alpha}$ near regular points (for some small $\alpha > 0$).

Finally, once we know that the free boundary is $C^{1,\alpha}$, we will bootstrap the regularity to C^∞. Once this was done, by Schauder estimates and a bootstrap argument we saw that solutions are actually C^∞.

Thus, how can we classify blow-ups? Do we get any extra information on u_0 that we did not have for u? (Otherwise it seems hopeless!)

The answer is *yes*: CONVEXITY. We will prove that all blow-ups are always *convex.* This is a huge improvement, since this yields that the contact set $\{u_0 = 0\}$ is also convex. Prior to that, we will also show that blow-ups are also *homogeneous.*

So, before the blow-up we had no information on the set $\{u = 0\}$, but after the blow-up we get that $\{u_0 = 0\}$ *is a convex cone.* Thanks to this we will be able to classify blow-ups, and thus to prove the regularity of the free boundary.

The main steps in the proof of the regularity of the free boundary will be the following:

1. $0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2$

2. Blow-ups u_0 are *homogeneous* and *convex.*

3. If the contact set has *positive density* at $x_\circ$, then $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$.

4. Deduce that the free boundary is $C^{1,\alpha}$ near $x_\circ$.

5. Deduce that the free boundary is C^∞ near $x_\circ$.

The proof we will present here for the convexity of blow-ups is new, based on the fact that they are homogeneous. We refer to [Caf98], [PSU12], [Wei99], and [KN77], for different proofs of the classification of blow-ups and/or of the regularity of free boundaries.

3.2 Classification of blow-ups

The aim of this Section is to classify all possible blow-ups u_0. For this, we will first prove that blow-ups are homogeneous, then we will prove that they are convex, and finally we will establish their complete classification.

3.2.1 Homogeneity of blow-ups

We start by proving that blow-ups are homogeneous. This is not essential in the proof of the regularity of the free boundary (see [Caf98]), but it actually simplifies it.

Therefore, from now on we consider a solution u satisfying (see Figure 3.3):

$$\begin{aligned} & u \in C^{1,1}(B_1) \\ & u \geq 0 \quad \text{in } B_1 \\ & \Delta u = 1 \quad \text{in } \{u > 0\} \\ & 0 \text{ is a free boundary point.} \end{aligned} \tag{3.3}$$

We will prove all the results around the origin (without loss of generality).

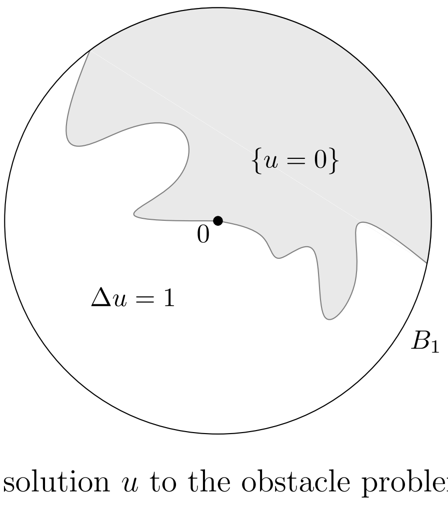

Figure 3.3: A solution u to the obstacle problem with $f \equiv 1$.

We will show that, for the original solution u in B_1, the closer we look at a free boundary point $x_\circ$, the closer is the solution to being homogeneous.

Proposition 3.1 (Homogeneity of blow-ups). *Let u be any solution to* (3.3)*. Then, any blow-up of u at* 0 *is homogeneous of degree* 2.

It is important to remark that not all global solutions to the obstacle problem in $\mathbb{R}^n$ are homogeneous. There exist global solutions u_0 that are convex, $C^{1,1}$, and whose contact set $\{u_0 = 0\}$ is an ellipsoid, for example. However, thanks to the previous result, we find that such non-homogeneous solutions cannot appear as blow-ups, i.e., that all blow-ups must be homogeneous.

We provide two different proofs of Proposition 3.1. The first one uses a monotonicity formula as introduced by Weiss; while the second one does not require any monotonicity formula and is due to Spruck.

For the first proof of Proposition 3.1, we need the following monotonicity formula due to Weiss [Wei99].

Theorem 3.2 (Weiss' monotonicity formula). *Let u be any solution to* (3.3)*. Then, the quantity*

$$W_u(r) := \frac{1}{r^{n+2}} \int_{B_r} \left\{\tfrac{1}{2}|\nabla u|^2 + u\right\} - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2 \tag{3.4}$$

is monotone in r, that is,

$$\frac{d}{dr} W_u(r) = \frac{1}{r^{n+4}} \int_{\partial B_r} (x \cdot \nabla u - 2u)^2 dx \geq 0$$

for $r \in (0, 1)$.

Proof. Let $u_r(x) = r^{-2} u(rx)$, and observe that

$$W_u(r) = \int_{B_1} \left\{\tfrac{1}{2}|\nabla u_r|^2 + u_r\right\} - \int_{\partial B_1} u_r^2.$$

Using this, together with

$$\frac{d}{dr}(\nabla u_r) = \nabla \frac{d}{dr} u_r,$$

we find

$$\frac{d}{dr} W_u(r) = \int_{B_1} \left\{\nabla u_r \cdot \nabla \frac{d}{dr} u_r + \frac{d}{dr} u_r\right\} - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r.$$

Now, integrating by parts we get

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = -\int_{B_1} \Delta u_r \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r.$$

Since $\Delta u_r = 1$ in $\{u_r > 0\}$ and $\frac{d}{dr} u_r = 0$ in $\{u_r = 0\}$, we have

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = -\int_{B_1} \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r.$$

Thus, we deduce

$$\frac{d}{dr} W_u(r) = \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r.$$

Using that on ∂B_1 we have $\partial_\nu = x \cdot \nabla$, combined with

$$\frac{d}{dr} u_r = \frac{1}{r} \{x \cdot \nabla u_r - 2u_r\}$$

yields

$$\frac{d}{dr} W_u(r) = \frac{1}{r} \int_{\partial B_1} (x \cdot \nabla u_r - 2u_r)^2 ,$$

which gives the desired result. □

We now give the:

First proof of Proposition 3.1. Let $u_r(x) = r^{-2} u(rx)$, and notice that we have the scaling property

$$W_{u_r}(\rho) = W_u(\rho r),$$

for any $r, \rho > 0$.

If u_0 is any blow-up of u at 0 then there is a sequence $r_j \to 0$ satisfying $u_{r_j} \to u_0$ in $C^1_{\text{loc}}(\mathbb{R}^n)$. Thus, for any $\rho > 0$ we have

$$W_{u_0}(\rho) = \lim_{r_j \to 0} W_{u_{r_j}}(\rho) = \lim_{r_j \to 0} W_u(\rho r_j) = W_u(0^+).$$

Notice that the limit $W_u(0^+) := \lim_{r \to 0} W_u(r)$ exists by monotonicity of W and since $u \in C^{1,1}$ implies $W_u(r) \geq -C$ for all $r \geq 0$.

Hence, the function $W_{u_0}(\rho)$ is *constant* in ρ. However, by Theorem 3.2 this yields that $x \cdot \nabla u_0 - 2u_0 \equiv 0$ in $\mathbb{R}^n$, and therefore u_0 is homogeneous of degree 2. □

Remark 3.3. Here, we used that a C^1 function u_0 is 2-homogeneous (i.e. $u_0(\lambda x) = \lambda^2 u_0(x)$ for all $\lambda \in \mathbb{R}_+$) if and only if $x \cdot \nabla u_0 \equiv 2u_0$.
This is because $\partial_\lambda|_{\lambda=1} \left\{\lambda^{-2} u_0(\lambda x)\right\} = x \cdot \nabla u_0 - 2u_0$.

We present an alternative (and quite different) proof of the homogeneity of blow-ups. Such proof is due to Spruck [Spr83] and is not based on any monotonicity formula.

Second proof of Proposition 3.1. Let u_0 be a blow-up given by the limit along a sequence $r_k \downarrow 0$,

$$u_0(x) := \lim_{k \to \infty} r_k^{-2} u(r_k x).$$

By taking polar coordinates $(\varrho, \theta) \in [0, +\infty) \times \mathbb{S}^{n-1}$ with $x = \varrho\theta$, and by denoting $\tilde{u}_0(\varrho, \theta) = u_0(\varrho\theta) = u_0(x)$, we will prove that $u_0(x) = \varrho^2 \tilde{u}_0(1, \theta) = |x|^2 u_0(x/|x|)$.

Let us define $\tau := -\log \varrho$, $\tilde{u}(\varrho, \theta) = u(x)$, and $\psi = \psi(\tau, \theta)$ as

$$\psi(\tau, \theta) := \varrho^{-2} \tilde{u}(\varrho, \theta) = e^{2\tau} u(e^{-\tau}\theta)$$

for $\tau \geq 0$. We observe that, since $\|u\|_{L^\infty(B_r)} \leq Cr^2$, ψ is bounded. Moreover, $\psi \in C^1((0, \infty) \times \mathbb{S}^{n-1}) \cap C^2(\{\psi > 0\})$ from the regularity of u; and $\partial_\tau \psi$ and $\nabla_\theta \psi$ are not only continuous, but also uniformly bounded in $[0, \infty) \times \mathbb{S}^{n-1}$. Indeed,

$$\left|\nabla_\theta \psi(\tau, \theta)\right| \leq e^\tau \left|\nabla u(e^{-\tau}\theta)\right| \leq C,$$

since $\|\nabla u\|_{L^\infty(B_r)} \leq Cr$ by $C^{1,1}$ regularity and the fact that $\nabla u(0) = 0$. For the same reason we also obtain

$$\left|\partial_\tau \psi(\tau, \theta)\right| \leq 2\psi(\tau, \theta) + e^\tau \left|\nabla u(e^{-\tau}\theta)\right| \leq C.$$

Observe that, by assumption, if we denote $\tau_k := -\log r_k$,

$$\psi(\tau_k, \theta) \to \tilde{u}_0(1, \theta) \quad \text{uniformly on } \mathbb{S}^{n-1}, \text{ as } k \to \infty. \tag{3.5}$$

Let us now write an equation for ψ. In order to do that, since we know that $\Delta u = \chi_{\{u>0\}}$ and $\chi_{\{u>0\}} = \chi_{\{\psi>0\}}$, we have

$$\Delta\left(\varrho^2 \psi(-\log \varrho, \theta)\right) = \chi_{\{\psi>0\}}.$$

By expanding the Laplacian in polar coordinates, $\Delta = \partial_{\varrho\varrho} + \frac{n-1}{\varrho}\partial_\varrho + \varrho^{-2}\Delta_{\mathbb{S}^{n-1}}$ (where $\Delta_{\mathbb{S}^{n-1}}$ denotes the spherical Laplacian, i.e. the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$) we obtain

$$2n\psi - (n+2)\partial_\tau \psi + \partial_{\tau\tau}\psi + \Delta_{\mathbb{S}^{n-1}}\psi = \chi_{\{\psi>0\}}. \tag{3.6}$$

We multiply the previous equality by $\partial_\tau \psi$, and integrate in $[0, \tau] \times \mathbb{S}^{n-1}$. We can consider the terms separately, integrating in τ first,

$$2n \int_{\mathbb{S}^{n-1}} \int_0^\tau \psi \partial_\tau \psi = n \int_{\mathbb{S}^{n-1}} \left(\psi^2(\tau, \theta) - \psi^2(0, \theta)\right) d\theta$$

and

$$\int_{\mathbb{S}^{n-1}} \int_0^\tau \partial_{\tau\tau}\psi \partial_\tau \psi = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \left((\partial_\tau \psi)^2(\tau, \theta) - (\partial_\tau \psi)^2(0, \theta)\right) d\theta,$$

and then integrating by parts in θ first, to integrate in τ afterwards:

$$\begin{aligned}\int_0^\tau \int_{\mathbb{S}^{n-1}} \Delta_{\mathbb{S}^{n-1}}\psi \partial_\tau \psi &= -\frac{1}{2} \int_0^\tau \int_{\mathbb{S}^{n-1}} \partial_\tau |\nabla_\theta \psi|^2 \\ &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} \left(|\nabla_\theta \psi|^2(0, \theta) - |\nabla_\theta \psi|^2(\tau, \theta)\right) d\theta.\end{aligned}$$

Finally, since $\partial_\tau\psi = 0$ whenever $\psi = 0$, we have $\chi_{\{\psi>0\}}\partial_\tau\psi = \partial_\tau\psi$ and

$$\int_{\mathbb{S}^{n-1}}\int_0^\tau \chi_{\{\psi>0\}}\partial_\tau\psi = \int_{\mathbb{S}^{n-1}} \big(\psi(\tau,\theta)-\psi(0,\theta)\big)\,d\theta.$$

In all, plugging back in (3.6) the previous expressions, and using that $\partial_\tau\psi$ and $\nabla_\theta\psi$ are uniformly bounded in $[0,\infty)\times\mathbb{S}^{n-1}$, we deduce that

$$\int_0^\infty\int_{\mathbb{S}^{n-1}}(\partial_\tau\psi)^2 = \int_0^\infty \|\partial_\tau\psi\|^2_{L^2(\mathbb{S}^{n-1})} \le C < \infty. \tag{3.7}$$

To finish, now observe that for any $|s| \le C_*$ fixed and for a sufficiently large k (such that $\tau_k + s \ge 0$),

$$\begin{aligned}\|\psi(\tau_k+s,\cdot)-\tilde{u}_0(1,\cdot)\|_{L^2(\mathbb{S}^{n-1})} &\le \|\psi(\tau_k+s,\cdot)-\psi(\tau_k,\cdot)\|_{L^2(\mathbb{S}^{n-1})}\\ &\quad+\|\psi(\tau_k,\cdot)-\tilde{u}_0(1,\cdot)\|_{L^2(\mathbb{S}^{n-1})}.\end{aligned}$$

The last term goes to zero, by (3.5). On the other hand, for the first term and by Hölder's inequality

$$\begin{aligned}\|\psi(\tau_k+s,\cdot)-\psi(\tau_k,\cdot)\|^2_{L^2(\mathbb{S}^{n-1})} &\le \left\|\int_0^s \partial_\tau\psi(\tau_k+\tau,\cdot)\,d\tau\right\|^2_{L^2(\mathbb{S}^{n-1})}\\ &\le C_*\left|\int_{\tau_k}^{\tau_k+s}\|\partial_\tau\psi\|^2_{L^2(\mathbb{S}^{n-1})}\right| \to 0,\end{aligned}$$

as $k\to\infty$, where we are using (3.7). Hence, $\psi(\tau_k+s,\cdot)\to\tilde{u}_0(1,\cdot)$ in $L^2(\mathbb{S}^{n-1})$ as $k\to\infty$, for any fixed $s\in\mathbb{R}$. On the other hand,

$$\psi(\tau_k+s,\theta) = e^{2s}r_k^{-2}u(e^{-2}r_k\theta) \to e^{2s}u_0(e^{-s}\theta) = e^{2s}\tilde{u}_0(e^{-s},\theta).$$

That is, for any $\rho = e^{-s} > 0$,

$$\tilde{u}_0(1,\cdot) = \rho^{-2}\tilde{u}_0(\rho,\theta),$$

as we wanted to see. □

By taking advantage of the fact that we know that blow-ups are 2-homogeneous, we can now give a short (and new) proof of the fact that they are also convex. More precisely, we will prove that 2-homogeneous global solutions to the obstacle problem are convex (and in particular, by Proposition 3.1, blow-ups are convex).

Theorem 3.4. *Let $u_0 \in C^{1,1}$ be any 2-homogeneous global solution to*

$$\begin{cases} u_0 \ \ge\ 0 & \text{in } \mathbb{R}^n\\ \Delta u_0 \ =\ 1 & \text{in } \{u_0>0\}\\ 0 \text{ is a free boundary point.} \end{cases}$$

Then, u_0 is convex.

The heuristic idea behind the proof of the previous result is the following: second derivatives D^2u_0 are harmonic in $\{u_0 > 0\}$ and satisfy that $D^2u_0 \geq 0$ on $\partial\{u_0 > 0\}$ (since $u_0 \geq 0$, it is “convex at the free boundary”). Since D^2u_0 is also 0-homogeneous, we can apply the maximum principle and conclude that $D^2u_0 \geq 0$ everywhere. That is, u_0 is convex. Let us formalize the previous heuristic idea into an actual proof.

We state a short lemma before providing the proof, which says that if $w \geq 0$ is superharmonic in $\{w > 0\}$, then it is superharmonic everywhere. For the sake of generality, we state the lemma for general H^1 functions, but we will use it only for functions that are also continuous.

Lemma 3.5. *Let $\Lambda \subset B_1$ be closed. Let $w \in H^1(B_1)$ be such that $w \geq 0$ on Λ and such that w is superharmonic in the weak sense in $B_1 \setminus \Lambda$. Then* $\min\{w, 0\}$ *is superharmonic in the weak sense in B_1.*

We now give the:

Proof of Theorem 3.4. Let $e \in \mathbb{S}^{n-1}$ and consider the second derivatives $\partial_{ee}u_0$. We define

$$w_0 := \min\{\partial_{ee}u_0, 0\}$$

and we claim that w_0 is superharmonic in $\mathbb{R}^n$, in the sense (**??**).

Indeed, let $\delta_t^2 u_0(x)$ for $t > 0$ be defined by

$$\delta_t^2 u_0(x) := \frac{u_0(x + te) + u_0(x - te) - 2u_0(x)}{t^2}.$$

Now, since $\Delta u_0 = \chi_{\{u_0>0\}}$, we have that

$$\Delta\delta_t^2 u_0 = \frac{1}{t^2}\big(\chi_{\{u_0(\,\cdot\,+te)\}} + \chi_{\{u_0(\,\cdot\,-te)\}} - 2\big) \leq 0 \quad \text{in} \quad \{u_0 > 0\}$$

in the weak sense. On the other hand, $\delta_t^2 u_0 \geq 0$ in $\{u_0 = 0\}$ and $\delta_t^2 u_0 \in C^{1,1}$. Thus, by Lemma 3.5, $w_t := \min\{\delta_t^2 u_0, 0\}$ is weakly superharmonic. Also notice that $\delta_t^2 u_0(x)$ is uniformly bounded independently of t, since $u_0 \in C^{1,1}$, and therefore w_t is uniformly bounded in t and converges pointwise to w_0 as $t \downarrow 0$. In particular, we have that w_0 is superharmonic.

Up to changing it in a set of measure 0, w_0 is lower semi-continuous. In particular, since w_0 is 0-homogeneous, it must attain its minimum at a point $y_\circ \in B_1$. But since $\fint_{B_r(y_\circ)} w_0$ is non-increasing for $r > 0$, we must have that w_0 is constant. Since it vanishes on the free boundary, we have $w_0 \equiv 0$. That is, for any $e \in \mathbb{S}^{n-1}$ we have that $\partial_{ee}u_0 \geq 0$ and therefore u_0 is convex.

□

3.2.2 Classification of blow-ups

We next want to classify all possible blow-ups for solutions to the obstacle problem (3.3). First, we will prove the following.

Proposition 3.6. *Let u be any solution to* (3.3)*, and let*

$$u_r(x) := \frac{u(rx)}{r^2}.$$

Then, for any sequence $r_k \to 0$ there is a subsequence $r_{k_j} \to 0$ such that

$$u_{r_{k_j}} \longrightarrow u_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^n)$$

as $k_j \to \infty$, for some function u_0 satisfying

$$\begin{cases} u_0 \in C^{1,1}_{\text{loc}}(\mathbb{R}^n) \\ u_0 \geq 0 \quad \text{in } B_1 \\ \Delta u_0 = 1 \quad \text{in } \{u_0 > 0\} \\ 0 \text{ is a free boundary point} \\ u_0 \text{ is convex} \\ u_0 \text{ is homogeneous of degree 2.} \end{cases}$$

Proof. By $C^{1,1}$ regularity of u, and by nondegeneracy, we have that

$$\frac{1}{C} \leq \sup_{B_1} u_r \leq C$$

for some $C > 0$. Moreover, again by $C^{1,1}$ regularity of u, we have

$$\|D^2 u_r\|_{L^\infty(B_{1/(2r)})} \leq C.$$

Since the sequence $\{u_{r_k}\}$, for $r_k \to 0$, is uniformly bounded in $C^{1,1}(K)$ for each compact set $K \subset \mathbb{R}^n$, there is a subsequence $r_{k_j} \to 0$ such that

$$u_{r_{k_j}} \longrightarrow u_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^n)$$

for some $u_0 \in C^{1,1}(K)$. Moreover, such function u_0 satisfies $\|D^2 u_0\|_{L^\infty(K)} \leq C$, with C independent of K, and clearly $u_0 \geq 0$ in K.

The fact that $\Delta u_0 = 1$ in $\{u_0 > 0\} \cap K$ can be checked as follows. For any smooth function $\eta \in C^\infty_c(\{u_0 > 0\} \cap K)$ we will have that, for k_j large enough, $u_{r_{k_j}} > 0$ in the support of η, and thus

$$\int_{\mathbb{R}^n} \nabla u_{r_{k_j}} \cdot \nabla \eta \, dx = -\int_{\mathbb{R}^n} \eta \, dx.$$

Since $u_{r_{k_j}} \to u_0$ in $C^1(K)$, we can take the limit $k_j \to \infty$ to get

$$\int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta \, dx = -\int_{\mathbb{R}^n} \eta \, dx.$$

Since this can be done for any $\eta \in C_c^\infty(\{u > 0\} \cap K)$, and for every $K \subset \mathbb{R}^n$, it follows that $\Delta u_0 = 1$ in $\{u_0 > 0\}$.

The fact that 0 is a free boundary point for u_0 follows simply by taking limits to $u_{r_{k_j}}(0) = 0$ and $\|u_{r_{k_j}}\|_{L^\infty(B_\rho)} \approx \rho^2$ for all $\rho \in (0,1)$. Finally, the homogeneity and convexity of u_0 follow from Proposition 3.1 and Theorem 3.4. □

Our next goal is to prove the following.

Theorem 3.7 (Classification of blow-ups)**.** *Let u be any solution to* (3.3)*, and let u_0 be any blow-up of u at 0. Then,*

(a) either

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2$$

for some $e \in \mathbb{S}^{n-1}$.

(b) or

$$u_0(x) = \frac{1}{2} x^T A x$$

for some matrix $A \geq 0$ with $\operatorname{tr} A = 1$.

It is important to remark here that, a priori, different subsequences could lead to different blow-ups u_0.

In order to establish Theorem 3.7, we will need the following.

Lemma 3.8. *Let $\Sigma \subset \mathbb{R}^n$ be any closed convex cone with nonempty interior, and with vertex at the origin. Let $w \in C(\mathbb{R}^n)$ be a function satisfying $\Delta w = 0$ in Σ^c, $w > 0$ in Σ^c, and $w = 0$ in Σ.*

Assume in addition that w is homogeneous of degree 1. Then, Σ must be a half-space.

Proof. By convexity of Σ, there exists a half-space $H = \{x \cdot e > 0\}$, with $e \in \mathbb{S}^{n-1}$, such that $H \subset \Sigma^c$.

Let $v(x) = (x \cdot e)_+$, which is harmonic and positive in H, and vanishes in H^c. By the Hopf Lemma (see Lemma 1.9), we have that $w \geq c_\circ d_\Sigma$ in $\Sigma^c \cap B_1$, where $d_\Sigma(x) = \operatorname{dist}(x, \Sigma)$ and $c_\circ$ is a small positive constant. In particular, since both w and d_Σ are homogeneous of degree 1, we deduce that $w \geq c_\circ d_\Sigma$ in all of Σ^c. Notice that, in order to apply the Hopf Lemma, we used that — by convexity of Σ — the domain Σ^c satisfies the interior ball condition.

Thus, since $d_\Sigma \geq d_{H^c} = v$, we deduce that $w \geq c_\circ v$, for some $c_\circ > 0$. The idea is now to consider the functions w and cv, and let $c > 0$ increase until the two functions touch at one point, which will give us a contradiction (recall that two harmonic functions cannot touch at an interior point). To do this rigorously, define

$$c_* := \sup\{c > 0 \,:\, w \geq cv \quad \text{in} \quad \Sigma^c\}.$$

Notice that $c_* \geq c_\circ > 0$. Then, we consider the function $w - c_* v \geq 0$. Assume that $w - c_* v$ is not identically zero. Such function is harmonic in H and hence, by the strict maximum principle, $w - c_* v > 0$ in H. Then, using the Hopf Lemma in H (see Lemma 1.9) we deduce that $w - c_* v \geq c_\circ d_{H^c} = c_\circ v$, since v is exactly the distance to H^c. But then we get that $w - (c_* + c_\circ)v \geq 0$, a contradiction with the definition of c_*.

Therefore, it must be $w - c_* v \equiv 0$. This means that w is a multiple of v, and therefore $\Sigma = H^c$, a half-space. □

We will also need the following.

Lemma 3.9. *Assume that $\Delta u = 1$ in $\mathbb{R}^n \setminus \partial H$, where ∂H is a hyperplane. If $u \in C^1(\mathbb{R}^n)$, then $\Delta u = 1$ in $\mathbb{R}^n$.*

Proof. Assume $\partial H = \{x_1 = 0\}$. For any ball $B_R \subset \mathbb{R}^n$, we consider the solution to $\Delta w = 1$ in B_R, $w = u$ on ∂B_R, and define $v = u - w$. Then, we have $\Delta v = 0$ in $B_R \setminus \partial H$, and $v = 0$ on ∂B_R. We want to show that u coincides with w, that is, $v \equiv 0$ in B_R.

For this, notice that since v is bounded, for $\kappa > 0$ large enough we have

$$v(x) \leq \kappa(2R - |x_1|) \quad \text{in} \quad B_R,$$

where $2R - |x_1|$ is positive in B_R and harmonic in $B_R \setminus \{x_1 = 0\}$. Thus, we may consider $\kappa^* := \inf\{\kappa \geq 0 : v(x) \leq \kappa(2R - |x_1|) \quad \text{in} \quad B_R\}$. Assume $\kappa^* > 0$. Since v and $2R - |x_1|$ are continuous in B_R, and $v = 0$ on ∂B_R, we must have a point $p \in B_R$ at which $v(p) = \kappa^*(2R - |p_1|)$. Moreover, since v is C^1, and the function $2R - |x_1|$ has a wedge on $\partial H = \{x_1 = 0\}$, we must have $p \in B_R \setminus \partial H$. However, this is not possible, as two harmonic functions cannot touch tangentially at an interior point p. This means that $\kappa^* = 0$, and hence $v \leq 0$ in B_R. Repeating the same argument with $-v$ instead of v, we deduce that $v \equiv 0$ in B_R, and thus the lemma is proved. □

Finally, we will use the following basic property of convex functions.

Lemma 3.10. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that the set $\{u = 0\}$ contains the straight line $\{te' : t \in \mathbb{R}\}$, $e' \in \mathbb{S}^{n-1}$. Then, $u(x + te') = u(x)$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.*

Proof. After a rotation, we may assume $e' = e_n$. Then, writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have that $u(0, x_n) = 0$ for all $x_n \in \mathbb{R}$, and we want to prove that $u(x', x_n) = u(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$ and all $x_n \in \mathbb{R}$.

Now, by convexity, given x' and x_n, for every $\varepsilon > 0$ and $M \in \mathbb{R}$ we have

$$(1 - \varepsilon)u(x', x_n) + \varepsilon u(0, x_n + M) \geq u((1 - \varepsilon)x', x_n + \varepsilon M).$$

Since $u(0, x_n + M) = 0$, choosing $M = \lambda/\varepsilon$ and letting $\varepsilon \to 0$ we deduce that

$$u(x', x_n) \geq u(x', x_n + \lambda).$$

Since this can be done for any $\lambda \in \mathbb{R}$ and $x_n \in \mathbb{R}$, the result follows. □

We finally establish the classification of blow-ups at regular points.

Proof of Theorem 3.7. Let u_0 be any blow-up of u at 0. We already proved that u_0 is convex and homogeneous of degree 2. We divide the proof into two cases.

Case 1. Assume that $\{u_0 = 0\}$ has nonempty interior. Then, we have $\{u_0 = 0\} = \Sigma$, a closed convex cone with nonempty interior.

For any direction $\tau \in \mathbb{S}^{n-1}$ such that $-\tau \in \mathring{\Sigma}$, we claim that

$$\partial_\tau u_0 \geq 0 \quad \text{in} \quad \mathbb{R}^n.$$

Indeed, for every $x \in \mathbb{R}^n$ we have that $u_0(x + \tau t)$ is zero for $t \ll -1$, and therefore by convexity of u_0 we get that $\partial_t u_0(x + \tau t)$ is monotone non-decreasing in t, and zero for $t \ll -1$. This means that $\partial_t u_0 \geq 0$, and thus $\partial_\tau u_0 \geq 0$ in $\mathbb{R}^n$, as claimed.

Now, for any such τ, we define $w := \partial_\tau u_0 \geq 0$. Notice that, at least for some $\tau \in \mathbb{S}^{n-1}$ with $-\tau \in \mathring{\Sigma}$, the function w is not identically zero. Moreover, since it is harmonic in Σ^c — recall that $\Delta u_0 = 1$ in Σ^c — then $w > 0$ in Σ^c.

But then, since w is homogeneous of degree 1, we can apply Lemma 3.8 to deduce that we must necessarily have that Σ is a half-space.

By convexity of u_0 and Lemma 3.10, this means that u_0 is a one-dimensional function, i.e., $u_0(x) = U(x \cdot e)$ for some $U : \mathbb{R} \to \mathbb{R}$ and some $e \in \mathbb{S}^{n-1}$. Thus, we have that $U \in C^{1,1}$ solves $U''(t) = 1$ for $t > 0$, with $U(t) = 0$ for $t \leq 0$. We deduce that $U(t) = \frac{1}{2}t_+^2$, and therefore $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$.

Case 2. Assume now that $\{u_0 = 0\}$ has empty interior. Then, by convexity, $\{u_0 = 0\}$ is contained in a hyperplane ∂H. Hence, $\Delta u_0 = 1$ in $\mathbb{R}^n \setminus \partial H$, with ∂H being a hyperplane, and $u_0 \in C^{1,1}$. It follows from Lemma 3.9 that $\Delta u_0 = 1$ in all of $\mathbb{R}^n$. But then all second derivatives of u_0 are harmonic and globally bounded in $\mathbb{R}^n$, so they must be constant. Hence, u_0 is a quadratic polynomial. Finally, since $u_0(0) = 0$, $\nabla u_0(0) = 0$, and $u_0 \geq 0$, we deduce that $u_0(x) = \frac{1}{2}x^T Ax$ for some $A \geq 0$, and since $\Delta u_0 = 1$, we have $\operatorname{tr} A = 1$. □

3.3 Regularity of the free boundary

The aim of this Section is to prove Theorem 3.19 below, i.e., that if u is any solution to (3.3) satisfying

$$\limsup_{r \to 0} \frac{\big|\{u = 0\} \cap B_r\big|}{|B_r|} > 0 \tag{3.8}$$

(i.e., the contact set has positive density at the origin), then the free boundary $\partial\{u > 0\}$ is C^∞ in a neighborhood of the origin.

For this, we will use the classification of blow-ups established in the previous Section.

$C^{1,\alpha}$ regularity of the free boundary

The first step here is to transfer the local information on u given by (3.8) into a blow-up u_0. More precisely, we next show that

$$(3.8) \quad \Longrightarrow \quad \begin{array}{c}\text{The contact set of a blow-up } u_0\\ \text{has nonempty interior.}\end{array}$$

Lemma 3.11. *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Then, there is at least one blow-up u_0 of u at 0 such that the contact set $\{u_0 = 0\}$ has nonempty interior.*

Proof. Let $r_k \to 0$ be a sequence along which

$$\lim_{r_k \to 0} \frac{\big|\{u = 0\} \cap B_{r_k}\big|}{|B_{r_k}|} \geq \theta > 0.$$

Such sequence exists (with $\theta > 0$ small enough) by assumption (3.8).

Recall that, thanks to Proposition 3.6, there exists a subsequence $r_{k_j} \downarrow 0$ along which $u_{r_{k_j}} \to u_0$ uniformly on compact sets of $\mathbb{R}^n$, where $u_r(x) = r^{-2}u(rx)$ and u_0 is convex.

Assume by contradiction that $\{u_0 = 0\}$ has empty interior. Then, by convexity, we have that $\{u_0 = 0\}$ is contained in a hyperplane, say $\{u_0 = 0\} \subset \{x_1 = 0\}$.

Since $u_0 > 0$ in $\{x_1 \neq 0\}$ and u_0 is continuous, we have that for each $\delta > 0$

$$u_0 \geq \varepsilon > 0 \quad \text{in } \{|x_1| > \delta\} \cap B_1$$

for some $\varepsilon > 0$.

Therefore, by uniform convergence of $u_{r_{k_j}}$ to u_0 in B_1, there is $r_{k_j} > 0$ small enough such that

$$u_{r_{k_j}} \geq \frac{\varepsilon}{2} > 0 \quad \text{in } \{|x_1| > \delta\} \cap B_1.$$

In particular, the contact set of $u_{r_{k_j}}$ is contained in $\{|x_1| \leq \delta\} \cap B_1$, so

$$\frac{\big|\{u_{r_{k_j}} = 0\} \cap B_1\big|}{|B_1|} \leq \frac{\big|\{|x_1| \leq \delta\} \cap B_1\big|}{|B_1|} \leq C\delta.$$

Rescaling back to u, we find

$$\frac{\big|\{u = 0\} \cap B_{r_{k_j}}\big|}{|B_{r_{k_j}}|} = \frac{\big|\{u_{r_{k_j}} = 0\} \cap B_1\big|}{|B_1|} < C\delta.$$

Since we can do this for every $\delta > 0$, we find that $\lim_{r_{k_j} \to 0} \frac{|\{u=0\} \cap B_{r_{k_j}}|}{|B_{r_{k_j}}|} = 0$, a contradiction. Thus, the lemma is proved. □

Combining the previous lemma with the classification of blow-ups from the previous Section, we deduce:

Corollary 3.12. *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Then, there is at least one blow-up of u at 0 of the form*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2, \qquad e \in \mathbb{S}^{n-1}.$$

Proof. The result follows from Lemma 3.11 and Theorem 3.7. □

We now want to use this information to show that the free boundary must be smooth in a neighborhood of 0. For this, we start with the following.

Proposition 3.13. *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Fix any $\varepsilon > 0$. Then, there exist $e \in \mathbb{S}^{n-1}$ and $r_\circ > 0$ such that*

$$\left|u_{r_\circ}(x) - \tfrac{1}{2}(x \cdot e)_+^2\right| \le \varepsilon \qquad \text{in} \quad B_1,$$

and

$$\left|\partial_\tau u_{r_\circ}(x) - (x \cdot e)_+(\tau \cdot e)\right| \le \varepsilon \qquad \text{in} \quad B_1$$

for all $\tau \in \mathbb{S}^{n-1}$.

Proof. By Corollary 3.12 and Proposition 3.6, we know that there is a subsequence $r_j \to 0$ for which $u_{r_j} \to \frac{1}{2}(x \cdot e)_+^2$ in $C^1_{\rm loc}(\mathbb{R}^n)$, for some $e \in \mathbb{S}^{n-1}$. In particular, for every $\tau \in \mathbb{S}^{n-1}$ we have $u_{r_j} \to \frac{1}{2}(x \cdot e)_+^2$ and $\partial_\tau u_{r_j} \to \partial_\tau\left[\frac{1}{2}(x \cdot e)_+^2\right]$ uniformly in B_1.

This means that, given $\varepsilon > 0$, there exists $j_\circ$ such that

$$\left|u_{r_{j_\circ}}(x) - \tfrac{1}{2}(x \cdot e)_+^2\right| \le \varepsilon \qquad \text{in} \quad B_1,$$

and

$$\left|\partial_\tau u_{r_{j_\circ}}(x) - \partial_\tau\left[\tfrac{1}{2}(x \cdot e)_+^2\right]\right| \le \varepsilon \qquad \text{in} \quad B_1.$$

Since $\partial_\tau\left[\frac{1}{2}(x \cdot e)_+^2\right] = (x \cdot e)_+(\tau \cdot e)$, the proposition is proved. □

Now, notice that if $(\tau \cdot e) > 0$, then the derivatives $\partial_\tau u_0 = (x \cdot e)_+(\tau \cdot e)$ are *nonnegative*, and strictly positive in $\{x \cdot e > 0\}$.

We want to transfer this information to $u_{r_\circ}$, and prove that $\partial_\tau u_{r_\circ} \ge 0$ in B_1 for all $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \ge \frac{1}{2}$. For this, we need a lemma.

Lemma 3.14. *Let u be any solution to* (3.3)*, and consider $u_{r_\circ}(x) = r_\circ^{-2}u(r_\circ x)$ and $\Omega = \{u_{r_\circ} > 0\}$.*

Assume that a function $w \in C(B_1)$ satisfies:

(a) w is bounded and harmonic in $\Omega \cap B_1$.

(b) $w = 0$ on $\partial\Omega \cap B_1$.

(c) Denoting $N_\delta := \{x \in B_1 : \mathrm{dist}(x, \partial\Omega) < \delta\}$, we have

$$w \ge -c_1 \quad \text{in} \quad N_\delta \qquad \text{and} \qquad w \ge C_2 > 0 \quad \text{in} \quad \Omega \setminus N_\delta.$$

If c_1/C_2 is small enough, and $\delta > 0$ is small enough, then $w \geq 0$ in $B_{1/2} \cap \Omega$.

Proof. Notice that in $\Omega \setminus N_\delta$ we already know that $w > 0$. Let $y_\circ \in N_\delta \cap \Omega \cap B_{1/2}$, and assume by contradiction that $w(y_0) < 0$.

Consider, in $B_{1/4}(y_\circ)$, the function

$$v(x) = w(x) - \gamma \left\{ u_{r_\circ}(x) - \frac{1}{2n}|x - y_\circ|^2 \right\}.$$

Then, $\Delta v = 0$ in $B_{1/4}(y_\circ) \cap \Omega$, and $v(y_\circ) < 0$. Thus, v must have a negative minimum in $\partial\big(B_{1/4}(y_\circ) \cap \Omega\big)$.

However, if c_1/C_2 and δ are small enough, then we reach a contradiction as follows:

On $\partial\Omega$ we have $v \geq 0$. On $\partial B_{1/4}(y_\circ) \cap N_\delta$ we have

$$v \geq -c_1 - C_\circ \gamma \delta^2 + \frac{\gamma}{2n}\left(\frac{1}{4}\right)^2 \geq 0 \quad \text{on} \quad \partial B_{1/4}(y_\circ) \cap N_\delta.$$

On $\partial B_{1/4}(y_\circ) \cap \big(\Omega \setminus N_\delta\big)$ we have

$$v \geq C_2 - C_\circ \gamma \geq 0 \quad \text{on} \quad \partial B_{1/4}(y_\circ) \cap \big(\Omega \setminus N_\delta\big).$$

Here, we used that $\|u_{r_\circ}\|_{C^{1,1}(B_1)} \leq C_\circ$, and chose $C_\circ c_1 \leq \gamma \leq C_2/C_\circ$. □

Using the previous lemma, we can now show that there is a cone of directions τ in which the solution is monotone near the origin.

Proposition 3.15. *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Let $u_r(x) = r^{-2}u(rx)$. Then, there exist $r_\circ > 0$ and $e \in \mathbb{S}^{n-1}$ such that*

$$\partial_\tau u_{r_\circ} \geq 0 \quad \text{in} \quad B_{1/2}$$

for every $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \geq \frac{1}{2}$.

Proof. By Proposition 3.13, for any $\varepsilon > 0$ there exist $e \in \mathbb{S}^{n-1}$ and $r_\circ > 0$ such that

$$\left| u_{r_\circ}(x) - \tfrac{1}{2}(x \cdot e)_+^2 \right| \leq \varepsilon \qquad \text{in} \quad B_1 \tag{3.9}$$

and

$$\left| \partial_\tau u_{r_\circ}(x) - (x \cdot e)_+ (\tau \cdot e) \right| \leq \varepsilon \qquad \text{in} \quad B_1 \tag{3.10}$$

for all $\tau \in \mathbb{S}^{n-1}$.

We now want to use Lemma 3.14 to deduce that $\partial_\tau u_{r_\circ} \geq 0$ if $\tau \cdot e \geq \frac{1}{2}$. First, we claim that

$$\begin{aligned} u_{r_\circ} &> 0 \quad \text{in} \quad \{x \cdot e > C_\circ \sqrt{\varepsilon}\}, \\ u_{r_\circ} &= 0 \quad \text{in} \quad \{x \cdot e < -C_\circ \sqrt{\varepsilon}\}, \end{aligned} \tag{3.11}$$

and therefore the free boundary $\partial\Omega = \partial\{u_{r_\circ} > 0\}$ is contained in the strip $\{|x \cdot e| \le C_\circ\sqrt{\varepsilon}\}$, for some $C_\circ$ depending only on n (see Figure 3.4). To prove this, notice that if $x \cdot e > C_\circ\sqrt{\varepsilon}$ then

$$u_{r_\circ} > \frac{1}{2}(C_\circ\sqrt{\varepsilon})^2 - \varepsilon > 0,$$

while if there was a free boundary point $x_\circ$ in $\{x \cdot e < -C_\circ\varepsilon\}$ then by nondegeneracy we would get

$$\sup_{B_{C_\circ\sqrt{\varepsilon}}(x_\circ)} u_{r_\circ} \ge c(C_\circ\sqrt{\varepsilon})^2 > 2\varepsilon,$$

a contradiction with (3.9).

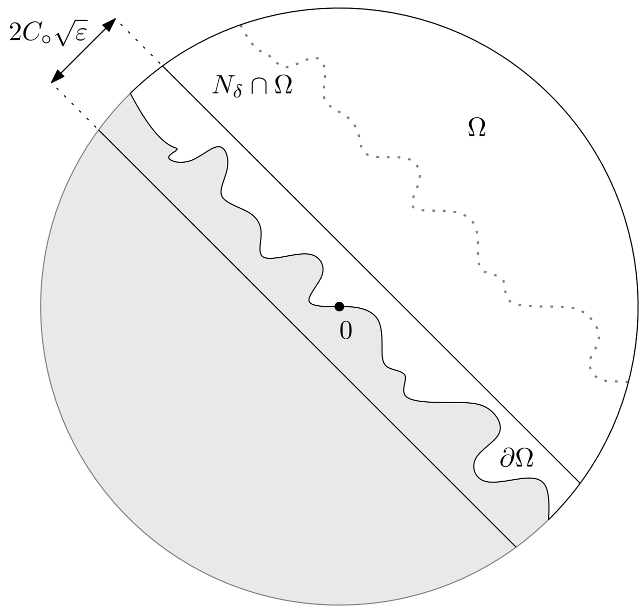

Figure 3.4: The setting in which we use Lemma 3.14.

Therefore, we have

$$\partial\Omega \subset \{|x \cdot e| \le C_\circ\sqrt{\varepsilon}\}.$$

Now, for each $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \ge \frac{1}{2}$ we define

$$w := \partial_\tau u_{r_\circ}.$$

In order to use Lemma 3.14, we notice:

(a) w is bounded and harmonic in $\Omega \cap B_1$.

(b) $w = 0$ on $\partial\Omega \cap B_1$.

(c) Thanks to (3.10), if $\delta \gg \sqrt{\varepsilon}$ then w satisfies

$$w \ge -\varepsilon \quad \text{in} \quad N_\delta$$

and

$$w \ge \delta/4 > 0 \quad \text{in} \quad (\Omega \setminus N_\delta) \cap B_1.$$

(We recall $N_\delta := \{x \in B_1 : \mathrm{dist}(x, \partial\Omega) < \delta\}$.)

Indeed, to check the last inequality we use that, by (3.11), we have $\{x \cdot e < \delta - C_\circ\sqrt{\varepsilon}\} \cap \Omega \subset N_\delta$. Thus, by (3.10), we get that for all $x \in (\Omega \setminus N_\delta) \cap B_1$

$$w \geq \frac{1}{2}(x \cdot e)_+ - \varepsilon \geq \frac{1}{2}\delta - \frac{1}{2}C_\circ\sqrt{\varepsilon} - \varepsilon \geq \frac{1}{4}\delta,$$

provided that $\delta \gg \sqrt{\varepsilon}$.

Using (a)-(b)-(c), we deduce from Lemma 3.14 that

$$w \geq 0 \quad \text{in} \quad B_{1/2}.$$

Since we can do this for every $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \frac{1}{2}$, the proposition is proved. □

As a consequence of the previous proposition, we find:

Corollary 3.16. *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Then, there exists $r_\circ > 0$ such that the free boundary $\partial\{u_{r_\circ} > 0\}$ is* Lipschitz *in $B_{1/2}$. In particular, the free boundary of u, $\partial\{u > 0\}$, is Lipschitz in $B_{r_\circ/2}$.*

Proof. This follows from the fact that $\partial_\tau u_{r_\circ} \geq 0$ in $B_{1/2}$ for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \frac{1}{2}$ (by Proposition 3.15), as explained next.

Let $x_\circ \in B_{1/2} \cap \partial\{u_{r_\circ} > 0\}$ be any free boundary point in $B_{1/2}$, and let

$$\Theta := \left\{\tau \in \mathbb{S}^{n-1} : \tau \cdot e > \tfrac{1}{2}\right\},$$

$$\Sigma_1 := \left\{x \in B_{1/2} : x = x_\circ - t\tau, \text{ with } \tau \in \Theta,\ t > 0\right\},$$

and

$$\Sigma_2 := \left\{x \in B_{1/2} : x = x_\circ + t\tau, \text{ with } \tau \in \Theta,\ t > 0\right\},$$

see Figure 3.5.

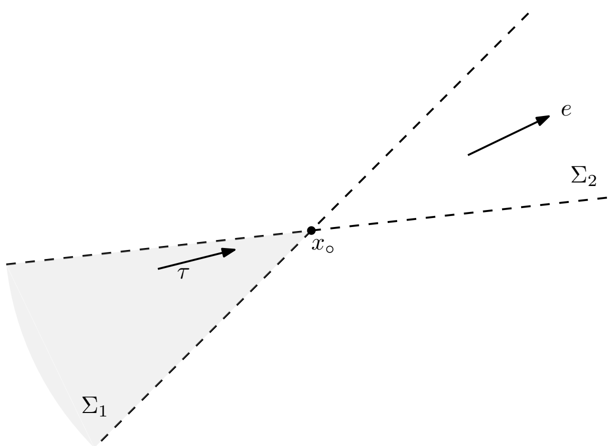

Figure 3.5: Representation of Σ_1 and Σ_2.

We claim that

$$\begin{cases} u_{r_\circ} = 0 & \text{in } \Sigma_1, \\ u_{r_\circ} > 0 & \text{in } \Sigma_2. \end{cases} \tag{3.12}$$

Indeed, since $u_{r_\circ}(x_\circ) = 0$, it follows from the monotonicity property $\partial_\tau u_{r_\circ} \geq 0$ — and the nonnegativity of $u_{r_\circ}$ — that $u_{r_\circ}(x_\circ - t\tau) = 0$ for all $t > 0$ and $\tau \in \Theta$. In particular, there cannot be any free boundary point in Σ_1.

On the other hand, by the same argument, if $u_{r_\circ}(x_1) = 0$ for some $x_1 \in \Sigma_2$ then we would have $u_{r_\circ} = 0$ in $\big\{x \in B_{1/2} : x = x_1 - t\tau, \text{ with } \tau \in \Theta,\ t > 0\big\} \ni x_\circ$, and in particular $x_\circ$ would not be a free boundary point. Thus, $u_{r_\circ}(x_1) > 0$ for all $x_1 \in \Sigma_2$, and (3.12) is proved.

Finally, notice that (3.12) yields that the free boundary $\partial\{u_{r_\circ} > 0\} \cap B_{1/2}$ satisfies both the interior and exterior cone condition, and thus it is Lipschitz. $\square$

Once we know that the free boundary is Lipschitz, we may assume without loss of generality that $e = e_n$ and that

$$\partial\{u_{r_\circ} > 0\} \cap B_{1/2} = \{x_n = g(x')\} \cap B_{1/2}$$

for a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$. Here, $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Now, we want to prove that Lipschitz free boundaries are $C^{1,\alpha}$. A key ingredient for this will be the following basic property of harmonic functions (see Figure 3.6 for a representation of the setting).

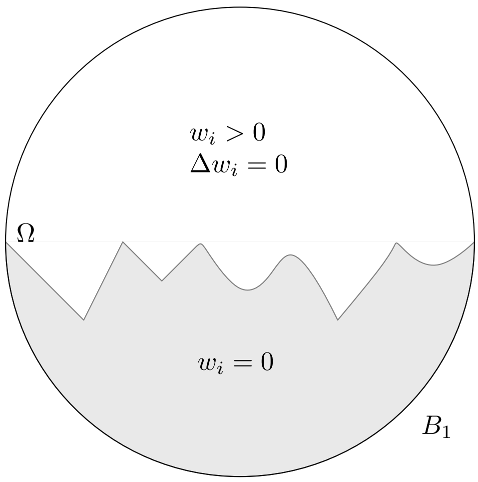

Figure 3.6: Setting of the boundary Harnack.

Theorem 3.17 (Boundary Harnack). *Let w_1 and w_2 be* positive harmonic *functions in $B_1 \cap \Omega$, where $\Omega \subset \mathbb{R}^n$ is any* Lipschitz domain.

Assume that w_1 and w_2 vanish on $\partial\Omega \cap B_1$, and $C_\circ^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_\circ$ for $i = 1, 2$. Then,

$$\frac{1}{C} w_2 \leq w_1 \leq C w_2 \qquad \text{in} \quad \overline{\Omega} \cap B_{1/2}.$$

Moreover,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C$$

for some small $\alpha > 0$. The constants α and C depend only on n, $C_\circ$, and Ω.

We refer to [DS20] for the boundary Harnack for more general operators and to [AS19, RT21] for the boundary Harnack for equations with a right hand side.

The boundary Harnack is a crucial tool in the study of free boundary problems, and in particular in the obstacle problem. Here, we use it to prove that the free boundary is $C^{1,\alpha}$ for some small $\alpha > 0$.

Proposition 3.18. *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Then, there exists $r_\circ > 0$ such that the free boundary $\partial\{u_{r_\circ} > 0\}$ is $C^{1,\alpha}$ in $B_{1/4}$, for some small $\alpha > 0$. In particular, the free boundary of u, $\partial\{u > 0\}$, is $C^{1,\alpha}$ in $B_{r_\circ/4}$.*

Proof. Let $\Omega = \{u_{r_\circ} > 0\}$. By Corollary 3.16, if $r_\circ > 0$ is small enough then (possibly after a rotation) we have

$$\Omega \cap B_{1/2} = \{x_n \geq g(x')\} \cap B_{1/2}$$

and the free boundary is given by

$$\partial\Omega \cap B_{1/2} = \{x_n = g(x')\} \cap B_{1/2},$$

where g is Lipschitz.

Let

$$w_2 := \partial_{e_n} u_{r_\circ}$$

and

$$w_1 := \partial_{e_i} u_{r_\circ} + \partial_{e_n} u_{r_\circ}, \qquad i = 1, ..., n-1.$$

Since $\partial_\tau u_{r_\circ} \geq 0$ in $B_{1/2}$ for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e_n \geq \frac{1}{2}$, we have that $w_2 \geq 0$ in $B_{1/2}$ and $w_1 \geq 0$ in $B_{1/2}$.

This is because $\partial_{e_i} + \partial_{e_n} = \partial_{e_i+e_n} = \sqrt{2}\partial_\tau$, with $\tau \cdot e_n = 1/\sqrt{2} > \frac{1}{2}$. Notice that we add the term $\partial_{e_n} u_{r_\circ}$ in w_1 in order to get a nonnegative function $w_2 \geq 0$.

Now since w_1 and w_2 are positive harmonic functions in $\Omega \cap B_{1/2}$, and vanish on $\partial\Omega \cap B_{1/2}$, we can use the boundary Harnack, Theorem 3.17, to get

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C$$

for some small $\alpha > 0$. Therefore, since $w_1/w_2 = 1 + \partial_{e_i} u_{r_\circ} / \partial_{e_n} u_{r_\circ}$, we deduce

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C. \tag{3.13}$$

Now, we claim that this implies that the free boundary is $C^{1,\alpha}$ in $B_{1/4}$. Indeed, if $u_{r_\circ}(x) = t$ then the normal vector to the level set $\{u_{r_\circ} = t\}$ is given by

$$\nu^i(x) = \frac{\partial_{e_i} u_{r_\circ}}{|\nabla u_{r_\circ}|} = \frac{\partial_{e_i} u_{r_\circ} / \partial_{e_n} u_{r_\circ}}{\sqrt{1 + \sum_{j=1}^{n-1} \left(\partial_{e_j} u_{r_\circ} / \partial_{e_n} u_{r_\circ}\right)^2}}, \qquad i = 1, ..., n.$$

This is a $C^{0,\alpha}$ function by (3.13), and therefore we can take $t \to 0$ to find that the free boundary is $C^{1,\alpha}$ (since the normal vector to the free boundary is given by a $C^{0,\alpha}$ function). □

So far we have proved that

$$\begin{pmatrix} \{u=0\} \text{ has positive} \\ \text{density at the origin} \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{any blow-up is} \\ u_0 = \frac{1}{2}(x\cdot e)_+^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{free boundary} \\ \text{is } C^{1,\alpha} \text{ near } 0 \end{pmatrix}$$

As a last step in this section, we will now prove that $C^{1,\alpha}$ free boundaries are actually C^∞.

3.3.1 Higher regularity of the free boundary

We want to finally prove the smoothness of free boundaries near regular points.

Theorem 3.19 (Smoothness of the free boundary near regular points)**.** *Let u be any solution to* (3.3)*, and assume that* (3.8) *holds. Then, the free boundary $\partial\{u>0\}$ is C^∞ in a neighborhood of the origin.*

For this, we need the following result.

Theorem 3.20 (Higher order boundary Harnack)**.** *Let $\Omega \subset \mathbb{R}^n$ be any $C^{k,\alpha}$ domain, with $k \geq 1$ and $\alpha \in (0,1)$. Let w_1, w_2 be two solutions of $\Delta w_i = 0$ in $B_1 \cap \Omega$, $w_i = 0$ on $\partial\Omega \cap B_1$, with $w_2 > 0$ in Ω.*

Assume that $C_\circ^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_\circ$. Then,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{k,\alpha}(\overline{\Omega}\cap B_{1/2})} \leq C,$$

where C depends only on n, k, α, $C_\circ$, and Ω.

We refer to [DS16] for the proof of such result.

Proof of Theorem 3.19. Let $u_{r_\circ}(x) = r_\circ^{-2} u(r_\circ x)$. By Proposition 3.18, we know that if $r_\circ > 0$ is small enough then the free boundary $\partial\{u_{r_\circ} > 0\}$ is $C^{1,\alpha}$ in B_1, and (possibly after a rotation) $\partial_{e_n} u_{r_\circ} > 0$ in $\{u_{r_\circ} > 0\} \cap B_1$. Thus, using the higher order boundary Harnack (Theorem 3.20) with $w_1 = \partial_{e_i} u_{r_\circ}$ and $w_2 = \partial_{e_n} u_{r_\circ}$, we find that

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{1,\alpha}(\overline{\Omega}\cap B_{1/2})} \leq C.$$

Actually, by a simple covering argument we find that

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{1,\alpha}(\overline{\Omega}\cap B_{1-\delta})} \leq C_\delta \tag{3.14}$$

for any $\delta > 0$.

Now, as in the proof of Proposition 3.18, we notice that if $u_{r_\circ}(x) = t$ then the normal vector to the level set $\{u_{r_\circ} = t\}$ is given by

$$\nu^i(x) = \frac{\partial_{e_i} u_{r_\circ}}{|\nabla u_{r_\circ}|} = \frac{\partial_{e_i} u_{r_\circ}/\partial_{e_n} u_{r_\circ}}{\sqrt{1 + \sum_{j=1}^n \left(\partial_{e_j} u_{r_\circ}/\partial_{e_n} u_{r_\circ}\right)^2}}, \qquad i = 1, ..., n.$$

By (3.14), this is a $C^{1,\alpha}$ function in $B_{1-\delta}$ for any $\delta > 0$, and therefore we can take $t \to 0$ to find that the normal vector to the free boundary is $C^{1,\alpha}$ inside B_1. But this means that the free boundary is actually $C^{2,\alpha}$.

Repeating now the same argument, and using that the free boundary is $C^{2,\alpha}$ in $B_{1-\delta}$ for any $\delta > 0$, we find that

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{2,\alpha}(\overline{\Omega} \cap B_{1-\delta'})} \le C_{\delta'},$$

which yields that the normal vector is $C^{2,\alpha}$ and thus the free boundary is $C^{3,\alpha}$. Iterating this argument, we find that the free boundary $\partial\{u_{r_\circ} > 0\}$ is C^∞ inside B_1, and hence $\partial\{u > 0\}$ is C^∞ in a neighborhood of the origin. □

This completes the study of *regular* free boundary points. It remains to understand what happens at points where the contact set has *density zero*.

Chapter 4

Singular points

We finally study the behavior of the free boundary at singular points, i.e., when

$$\lim_{r\to 0} \frac{\big|\{u=0\}\cap B_r\big|}{|B_r|} = 0. \tag{4.1}$$

For this, we first notice that, as a consequence of the results of the previous Section, we get the following.

Proposition 4.1. *Let u be any solution to* (3.3). *Then, we have the following dichotomy:*

(a) Either (3.8) *holds and all blow-ups of u at* 0 *are of the form*

$$u_0(x) = \frac{1}{2}(x\cdot e)_+^2,$$

for some $e \in \mathbb{S}^{n-1}$.

(b) Or (4.1) *holds and all blow-ups of u at* 0 *are of the form*

$$u_0(x) = \frac{1}{2}x^T A x,$$

for some matrix $A \geq 0$ with $\operatorname{tr} A = 1$.

Points of type (a) were studied in the previous Section; they are called *regular* points and the free boundary is C^∞ around them (in particular, the blow-up is unique). Points of type (b) are those at which the contact set has zero density, and are called *singular* points.

To prove the result, we need the following:

Lemma 4.2. *Let u be any solution to* (3.3), *and assume that* (4.1) *holds. Then, every blow-up of u at* 0 *satisfies* $|\{u_0=0\}|=0$.

Proof. Let u_0 be a blow-up of u at 0, i.e., $u_{r_k} \to u_0$ in $C^1_{\rm loc}(\mathbb{R}^n)$ along a sequence $r_k \to 0$, where $u_r(x) = r^{-2}u(rx)$.

Notice that the functions u_r solve

$$\Delta u_r = \chi_{\{u_r>0\}} \quad \text{in} \quad B_1,$$

in the sense that

$$\int_{B_1} \nabla u_r \cdot \nabla \eta \, dx = \int_{B_1} \chi_{\{u_r>0\}} \eta \, dx \qquad \text{for all } \eta \in C^\infty_c(B_1). \tag{4.2}$$

Moreover, by assumption (4.1), we have $\big|\{u_r = 0\} \cap B_1\big| \longrightarrow 0$, and thus taking limits $r_k \to 0$ in (4.2) we deduce that $\Delta u_0 = 1$ in B_1. Since we know that u_0 is convex, nonnegative, and homogeneous, this implies that $|\{u_0 = 0\}| = 0$. □

We can now give the:

Proof of Theorem 4.1. By the classification of blow-ups (Theorem 3.7), the possible blow-ups can only have one of the two forms presented. If (3.8) holds for at least one blow-up, thanks to the smoothness of the free boundary (by Proposition 3.18), it holds for all blow-ups, and thus, by Corollary 3.12, $u_0(x) = \frac{1}{2}(x \cdot e)^2_+$ (and in fact, the smoothness of the free boundary yields uniqueness of the blow-up in this case).

If (4.1) holds, then by Lemma 4.2 the blow-up u_0 must satisfy $\big|\{u_0 = 0\}\big| = 0$, and thus we are in case (b) (see the proof of Theorem 3.7). □

In the previous Section we proved that the free boundary is C^∞ in a neighborhood of any regular point. A natural question then is to understand better the solution u near singular points.

One of the main results in this direction is the following.

Theorem 4.3 (Uniqueness of blow-ups at singular points)**.** *Let u be any solution to* (3.3)*, and assume that* 0 *is a singular free boundary point.*

Then, there exists a homogeneous quadratic polynomial $p_2(x) = \frac{1}{2}x^T Ax$, with $A \geq 0$ and $\Delta p_2 = 1$, such that

$$u_r \longrightarrow p_2 \qquad \textit{in} \quad C^1_{\rm loc}(\mathbb{R}^n).$$

In particular, the blow-up of u at 0 is unique, and $u(x) = p_2(x) + o(|x|^2)$.

To prove this, we need the following monotonicity formula due to Monneau.

Theorem 4.4 (Monneau's monotonicity formula)**.** *Let u be any solution to* (3.3)*, and assume that* 0 *is a singular free boundary point.*

Let q be any homogeneous quadratic polynomial with $q \geq 0$, $q(0) = 0$, and $\Delta q = 1$. Then, the quantity

$$M_{u,q}(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u-q)^2$$

is monotone in r, that is, $\frac{d}{dr}M_{u,q}(r) \geq 0$.

Proof. We sketch the argument here, and refer to [PSU12, Theorem 7.4] for more details.

We first notice that

$$M_{u,q}(r) = \int_{\partial B_1} \frac{(u-q)^2(rx)}{r^4},$$

and hence a direct computation yields

$$\frac{d}{dr} M_{u,q}(r) = \frac{2}{r^{n+4}} \int_{\partial B_r} (u-q)\left\{x \cdot \nabla(u-q) - 2(u-q)\right\}.$$

On the other hand, it turns out that

$$\frac{1}{r^{n+3}} \int_{\partial B_r} (u-q)\left\{x \cdot \nabla(u-q) - 2(u-q)\right\} = W_u(r) - W_u(0^+) + \\ + \frac{1}{r^{n+2}} \int_{B_r} (u-q)\Delta(u-q),$$

where $W_u(r)$ (as defined in (3.4)) is monotone increasing in $r > 0$ thanks to Theorem 3.2. Thus, we have

$$\frac{d}{dr} M_{u,q}(r) \geq \frac{2}{r^{n+3}} \int_{B_r} (u-q)\Delta(u-q).$$

But since $\Delta u = \Delta q = 1$ in $\{u > 0\}$, and $(u-q)\Delta(u-q) = q \geq 0$ in $\{u = 0\}$, we have

$$\frac{d}{dr} M_{u,q}(r) \geq \frac{2}{r^{n+3}} \int_{B_r \cap \{u=0\}} q \geq 0,$$

as wanted. □

We can now give the:

Proof of Theorem 4.3. By Proposition 4.1 (and Proposition 3.6), we know that at any singular point we have a subsequence $r_j \to 0$ along which $u_{r_j} \to p$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, where p is a 2-homogeneous quadratic polynomial satisfying $p(0) = 0$, $p \geq 0$, and $\Delta p = 1$. Thus, we can use Monneau's monotonicity formula with such polynomial p to find that

$$M_{u,p}(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u-p)^2$$

is monotone increasing in $r > 0$. In particular, the limit $\lim_{r\to 0} M_{u,p}(r) := M_{u,p}(0^+)$ exists.

Now, recall that we have a sequence $r_j \to 0$ along which $u_{r_j} \to p$. In particular, $r_j^{-2}\left\{u(r_j x) - p(r_j x)\right\} \longrightarrow 0$ locally uniformly in $\mathbb{R}^n$, i.e.,

$$\frac{1}{r_j^2} \|u - p\|_{L^\infty(B_{r_j})} \longrightarrow 0$$

as $r_j \to 0$. This yields that

$$M_{u,p}(r_j) \le \frac{1}{r_j^{n+3}} \int_{\partial B_{r_j}} \|u-p\|^2_{L^\infty(B_{r_j})} \longrightarrow 0$$

along the subsequence $r_j \to 0$, and therefore $M_{u,p}(0^+) = 0$.

Let us show that this implies the uniqueness of blow-ups. Indeed, if there was another subsequence $r_\ell \to 0$ along which $u_{r_\ell} \to q$ in $C^1_{\rm loc}(\mathbb{R}^n)$, for a 2-homogeneous quadratic polynomial q, then we would repeat the argument above to find that $M_{u,q}(0^+) = 0$. But then this yields, by homogeneity of p and q,

$$\int_{\partial B_1} (p-q)^2 = \frac{1}{r^{n+3}} \int_{\partial B_r} (p-q)^2 \le 2M_{u,p}(r) + 2M_{u,q}(r) \longrightarrow 0,$$

and hence

$$\int_{\partial B_1} (p-q)^2 = 0.$$

This means that $p = q$, and thus the blow-up of u at 0 is unique.

Let us finally show that $u(x) = p(x) + o(|x|^2)$, i.e., $r^{-2}\|u-p\|_{L^\infty(B_r)} \to 0$ as $r \to 0$. Indeed, assume by contradiction that there is a subsequence $r_k \to 0$ along which

$$r_k^{-2}\|u-p\|_{L^\infty(B_{r_k})} \ge c_1 > 0.$$

Then, there would be a subsequence of r_{k_i} along which $u_{r_{k_i}} \to u_0$ in $C^1_{\rm loc}(\mathbb{R}^n)$, for a certain blow-up u_0 satisfying $\|u_0 - p\|_{L^\infty(B_1)} \ge c_1 > 0$. However, by uniqueness of blow-ups it must be $u_0 = p$, and hence we reach a contradiction. □

Summarizing, we have proved the following result:

Theorem 4.5. *Let u be any solution to* (3.3). *Then, we have the following dichotomy:*

(a) Either all blow-ups of u at 0 are of the form

$$u_0(x) = \frac{1}{2}(x\cdot e)_+^2 \qquad \textit{for some} \quad e \in \mathbb{S}^{n-1},$$

and the free boundary is C^∞ in a neighborhood of the origin.

(b) Or there is a homogeneous quadratic polynomial p, with $p(0) = 0$, $p \ge 0$, and $\Delta p = 1$, such that

$$\|u-p\|_{L^\infty(B_r)} = o(r^2) \qquad \textit{as} \quad r \to 0.$$

In particular, when this happens we have

$$\lim_{r\to 0} \frac{\big|\{u=0\}\cap B_r\big|}{|B_r|} = 0.$$

The last question that remains to be answered is: How large can the set of singular points be? This is the topic of the following section.

4.1 On the size of the singular set

We finish this chapter with a discussion of more recent results (as well as some open problems) about the set of singular points.

Recall that a free boundary point $x_\circ \in \partial\{u > 0\}$ is singular whenever

$$\lim_{r\to 0} \frac{\big|\{u = 0\} \cap B_r(x_\circ)\big|}{|B_r(x_\circ)|} = 0.$$

The main known result on the size of the singular set reads as follows.

Theorem 4.6 ([Caf98]). *Let u be any solution to* (3.3). *Let $\Sigma \subset B_1$ be the set of singular points.*

Then, $\Sigma \cap B_{1/2}$ is locally contained in a C^1 manifold of dimension $n-1$.

This result is sharp, in the sense that it is not difficult to construct examples in which the singular set is $(n-1)$-dimensional; see [Sch77].

As explained below, such result essentially follows from the uniqueness of blow-ups at singular points, established in the previous section.

Indeed, given any singular point $x_\circ$, let $p_{x_\circ}$ be the blow-up of u at $x_\circ$ (recall that $p_{x_\circ}$ is a nonnegative 2-homogeneous polynomial). Let k be the dimension of the set $\{p_{x_\circ} = 0\}$ — notice that this is a proper linear subspace of $\mathbb{R}^n$, so that $k \in \{0, ..., n-1\}$ — and define

$$\Sigma_k := \Big\{x_\circ \in \Sigma : \dim(\{p_{x_\circ} = 0\}) = k\Big\}. \tag{4.3}$$

Clearly, $\Sigma = \bigcup_{k=0}^{n-1} \Sigma_k$.

The following result gives a more precise description of the singular set.

Proposition 4.7 ([Caf98]). *Let u be any solution to* (3.3). *Let $\Sigma_k \subset B_1$ be defined by* (4.3)*, $k = 1, ..., n-1$. Then, Σ_k is locally contained in a C^1 manifold of dimension k.*

4.1.1 Generic regularity

In PDE problems in which singularities may appear, it is very natural and important to understand whether these singularities appear "often", or if instead "most" solutions have no singularities.

In the context of the obstacle problem, the key question is to understand the generic regularity of free boundaries. Explicit examples show that singular points in the obstacle problem can form a very large set, of dimension $n-1$ (as large as the regular set). Still, singular points are expected to be rare (see [Sch74]):

Conjecture (Schaeffer, 1974): *Generically, the weak solution of the obstacle problem is also a strong solution, in the sense that the free boundary is a C^∞ manifold.*

In other words, the conjecture states that, generically, the free boundary has *no* singular points.

The first result in this direction was established by Monneau in 2003, who proved the following.

Theorem 4.8 ([Mon03]). *Schaeffer's conjecture holds in* $\mathbb{R}^2$.

More precisely, Monneau considers a 1-parameter family of solutions u_λ, with $\lambda \in (0,1)$, such that

$$\begin{cases} \Delta u_\lambda = \chi_{\{u_\lambda > 0\}} & \text{in } \Omega \\ u_\lambda = \lambda & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > 0$ on $\partial\Omega$.

Then, the first step is to notice that not only each of the singular sets $\Sigma_\lambda \subset \Omega$ is contained in a C^1 manifold of dimension $(n-1)$, but actually the union $\bigcup_{\lambda\in(0,1)} \Sigma_\lambda \subset \Omega$ is still contained in an $(n-1)$-dimensional manifold.

After that, we look at the free boundary as a set in $\Omega \times (0,1) \ni (x, \lambda)$, and notice that it can be written as a graph $\{\lambda = h(x)\}$, for some function h. A second key step in the proof is to show that h is Lipschitz and, furthermore, it has zero gradient at any singular point. This, combined with the coarea formula, yields that in $\mathbb{R}^2$ the set of singular points is empty for almost every $\lambda \in (0,1)$, which implies Theorem 4.8.

Finally, the best known result in this direction was established very recently by Figalli, Serra, and the second author.

Theorem 4.9 ([FRS20]). *Schaeffer's conjecture holds in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

The proof of this result is based on a new and very fine understanding of singular points. For this, [FRS20] combines Geometric Measure Theory tools, PDE estimates, several dimension reduction arguments, and even several new monotonicity formulas.

It remains an open problem to decide whether or not Schaeffer's conjecture holds in dimensions $n \geq 5$ or not.

In the next section we will give the proof of the Schaeffer's conjecture in $\mathbb{R}^2$ (Theorem 4.8) based on the paper of Monneau, see [Mon03].

Chapter 5

Schaeffer's conjecture

5.1 Monneau's proof of the Schaeffer's conjecture

From now on we are in the setting described in Theorem 4.8.

Let us introduce a new function:

$$h(x) := \sup\{\lambda \in [0, +\infty) \mid u_\lambda(x) = 0\}.$$

We will prove the next two propositions in order to complete the proof of the Theorem.

Proposition 5.1. *The function h is Lipschitz-continuous on $\overline{\Omega}$*

Proposition 5.2.

$$\begin{aligned} \{h \geq \lambda\} &= \{u_\lambda = 0\} \\ \{h = \lambda\} &= \partial\{u_\lambda = 0\} \end{aligned} \tag{5.1}$$

Monneau improved Caffarelli's Theorem 4.6 in the following sense.

Theorem 5.3 ([Mon03], A C^1-submanifolds contains almost all singulararities)**.** *Let*

$$S_\lambda = \textit{the set of singular points of the coincidence set of } u_\lambda$$

and

$$S := \bigcup_{\lambda > 0} S_\lambda$$

then there exists a set $E \subset S$ such that $\mathcal{H}^{n-1}(E) = 0$ and $S \setminus E$ is incuded in a C^1 $(n-1)$-dimensional submanifold of finite $(n-1)$-volume.

Let us now recall the coarea formula.

Theorem 5.4 ([Fed69] Coarea Formula)**.** *Let $A \subset \mathbb{R}^m$ a set such that $A \setminus B$ is included in a C^1 k-dimensional submanifold of finite k-volume and $\mathcal{H}^k(B) = 0$. Let a function*

$$f : A \to \mathbb{R}$$

wich is Lipschitz on A. Then the ∇f is defined $\mathcal{H}^k$ almost everywhere on A and

$$\int_A |\nabla f| \, d\mathcal{H}^k = \int_{\mathbb{R}} \mathcal{H}^{k-1}(f^{-1}(y))dy$$

Proposition 5.5 (The function h has null gradient on the singular set)**.**

$$\limsup_{|X'-X|\to 0} \frac{|h(X') - h(X)|}{|X' - X|} = 0 \quad \forall\ X,\ X' \in S.$$

So from the theorems and propositions above, we can deduce that

$$0 = \int_S |\nabla h| \, d\mathcal{H}^{n-1} = \int_0^{+\infty} \mathcal{H}^{n-2}((h|_S)^{-1}(\lambda)) \, d\lambda$$

with

$$S_\lambda = (h|_S)^{-1}(\lambda),$$

so if $n = 2$ we get that $\mathcal{H}^0(S) = 0$, and this conclude the proof of the Schaeffer's conjecture in $\mathbb{R}^2$.

For the proof of Proposition 5.5 we need the following Theorem due to Caffarelli, see [Caf98]

Definition 5.6 (Thickness of the Coincidence Set)**.** We define the thickness of the coincidence set $\{u = 0\}$ in a ball $B_r(X_0)$ by

$$\delta_r(X_0) = \frac{1}{r} \text{ m.d.} \left(\{u = 0\} \cap B_r(X_0)\right)$$

where the minimum diameter (m.d.) of $\{u = 0\} \cap B_r(X_0)$ is the infimum of the distances between pairs of parallel hyperplanes whose strip determined by them contains it.

Theorem 5.7 (Caffarelli's Geometric Criterion)**.** *For each $r > 0$, there exists a critical thickness $\sigma_0(r)$ with $\sigma_0(r) \to 0$ as $r \to 0$, such that if*

$$\delta_r(X_0) > \sigma_0(r)$$

for some point X_0 of the free boundary and for one radius $r > 0$, then the point X_0 is regular. Moreover, this function $\sigma_0(r)$ only depends on bounds on $\|D^2 u\|_{L^\infty(\Omega)}$ and $d(X_0, \partial\Omega)$.

Proof of Proposition 5.5. If the proposition is false, then there exists $\delta > 0$, and sequences of singular points $X_k, X'_k \in S$ such that

$$\frac{h(X'_k) - h(X_k)}{|X'_k - X_k|} \geq \delta > 0 \quad \text{and} \quad |X'_k - X_k| \to 0.$$

By equality (5.1) we have in particular

$$\{u_h(X_k) = 0\} \supset X'_k,$$

which (since h is Lipschitz) is improved in

$$\{u_h(X_k) = 0\} \supset B_{r_k}(X'_k) \quad \text{where} \quad r_k = \frac{\delta |X'_k - X_k|}{\mathrm{Lip}(h)}.$$

Let us introduce the following blow-up sequence:

$$u^k(X) = \frac{u_h(X_k)(X_k + r_k X)}{r_k^2}.$$

In particular,

$$u^k = 0 \quad \text{on the ball} \quad B_1(Y_k) \quad \text{with} \quad |Y_k| = \frac{\mathrm{Lip}(h)}{\delta}.$$

From the geometric criterion (Theorem 5.7), this gives a contradiction with the fact that 0 is a singular point for u^k.

□

We will now prove Proposition 5.2. We first need the following two lemmas.

Lemma 5.8.

$$\{u_\lambda = 0\} \subset \{u_{\lambda' = 0}\} \quad \textit{if } \lambda' \leq \lambda.$$

Proof. Apply the maximum principle to $u_\lambda - u_{\lambda'}$ □

Lemma 5.9. *For every point $X_0 \in \Omega$, $X_0 \in \partial\{u_{h(X_0)} = 0\}$*

Proof. This lemma is a consequence of the fact that for $\lambda_1 = h(X_0)$ and for every $s > 0$:

$$u_{\lambda_1 + s} > 0$$

and from the nondegeneracy we get

$$\sup_{B_r(X_0)} (u_{\lambda_1+s} - u_{\lambda_1+s}(X_0)) \geq \frac{r^2}{2n}$$

By continuity of the map $(X, \lambda) \mapsto u_\lambda(X)$, we get in $s = 0$:

$$\sup_{B_r(X_0)} u_{\lambda_1} \geq \frac{r^2}{2n}$$

wich proves the lemma. □

Proof of Proposition 5.2. From Lemma 5.9 and the continuity of the map $(X, \lambda) \mapsto u_\lambda(X)$, we get

$$\{u_\lambda = 0\} = \{h \geq \lambda\}.$$

To prove that

$$\partial\{u_\lambda = 0\} = \{h = \lambda\}$$

we only need (from Lemma 5.10) to prove that

$$\partial\{u_\lambda = 0\} \subset \{h = \lambda\}$$

which is a consequence of

$$\partial\{u_\lambda = 0\} \subset \{h \geq \lambda\}$$

and the fact that h is Lipschitz (Proposition 5.1) which avoids the values $h > \lambda$. This ends the proof of Proposition 5.2. □

Now we will prove that h is continuous, and after that, we prove that it is actually Lipschitz, but we need continuity first.

Proposition 5.10. *The function h is continuous.*

Proof. We will here introduce a perturbation argument which will insure easily the continuity of h. We will denote by $\eta > 0$ the parameter of the perturbation:

$$u_\lambda^\eta = (1 - \eta\lambda)u_\lambda.$$

For $\eta > 0$ small enough, the maximum principle implies

$$u_\lambda^\eta \leq u_{\lambda'}^\eta \quad \text{on } \Omega \quad \text{if } \lambda \leq \lambda'.$$

Now let us assume that h is not continuous in $X_0 \in \Omega$. Then there exist $\delta > 0$ and a sequence of points

$$X_k \to X_0 \quad \text{with} \quad |h(X_k) - h(X_0)| \geq \delta > 0.$$

If $h(X_k) - h(X_0) \leq -\delta$, then let

$$v_2^k(X) := (1 - \eta h(X_k)) \frac{u_{h(X_k)}(X_0 + |X_k - X_0|X)}{|X_k - X_0|^2} \leq$$

$$\leq v_1^k(X) := (1 - \eta h(X_0)) \frac{u_{h(X_0)}(X_0 + |X_k - X_0|X)}{|X_k - X_0|^2}.$$

Because by Lemma 5.10 X_0 is a point of the free boundary $\partial\{u_{h(X_0)} = 0\}$, by Classification of blow-ups we know that the blow-up limit v_1^0 satisfies with $\lambda_1 = h(X_0)$:

$$\frac{1}{1 - \eta\lambda_1} v_1^0(X) = \begin{cases} \frac{1}{2}\, tX \cdot Q_1 \cdot X \geq 0 \quad \text{with} \quad \text{tr } Q_1 = 1 \\ \text{or} \\ \frac{1}{2}(\max(\langle X, \nu_1\rangle, 0))^2. \end{cases}$$

The sequence v_2^0 has also a blow-up limit:

$$v_2^0 \quad \text{defined on } \mathbb{R}^n.$$

Then if we introduce a blow-down sequence with $\mu \to +\infty$:

$$v_2^{0\mu}(X) := \frac{v_2^0(\mu X)}{\mu^2} \leq v_1^{0\mu}(X) := v_1^0(X).$$

Also the blow-down limit v_2^∞ satisfies with $\lambda_2 = \lim h(X_k) \leq \lambda_1 - \delta$:

$$\frac{1}{1-\eta\lambda_2} v_2^\infty(X) = \begin{cases} \frac{1}{2} X \cdot Q_2 \cdot X \geq 0 \quad \text{with} \quad \text{tr } Q_2 = 1 \\ \text{or} \\ \frac{1}{2}(\max(\langle X, \nu_2 \rangle, 0))^2. \end{cases}$$

The fact that

$$1 - \eta\lambda_2 > 1 - \eta\lambda_1$$

gives a contradiction with the inequality

$$v_2^\infty \leq v_1^\infty := v_1^0 \quad \text{on } \mathbb{R}^n.$$

We get a similar contradiction with $h(X_k) - h(X_0) \geq \delta$. □

The last thing to prove in order to conclude the proof of the Schaeffer's conjecture in $\mathbb{R}^2$ is that h is Lipschitz.

Proposition 5.11. *The function h is Lipschitz.*

To prove that h is Lipschitz, we need to introduce the following family of functions for $\delta > 0$ and $\varepsilon > 0$:

$$v^\varepsilon(X) = \sup_{Y \in B_{\varepsilon\delta}(X)} u_{\lambda-\delta}(Y)$$

which are subsolutions (as it will be proved below) to the obstacle problem (1.4) on

$$\Omega_{(-\varepsilon\delta)} = \{X \in \Omega,\, d(X, \partial\Omega) > \varepsilon\delta\}.$$

For $\delta > 0$ fixed, by a continuity method varying the parameter $\varepsilon > 0$, we will prove that these subsolutions stay under the solution u_λ until some critical value $\varepsilon_c > 0$:

$$v^\varepsilon \leq u_\lambda \quad \text{on} \quad \Omega_{(-\varepsilon\delta)} \quad \text{for} \quad 0 \leq \varepsilon \leq \varepsilon_c := \frac{1}{|\nabla u_\lambda|_{L^\infty(\overline{\Omega}\setminus\Omega_{(-\varepsilon\delta)})}}. \tag{5.2}$$

For $\varepsilon = \varepsilon_c$, this implies

$$d(\{u_\lambda = 0\}, \{u_{\lambda-\delta} > 0\}) \geq \varepsilon_c \delta.$$

In particular, if $X' \in \partial\{u_\lambda = 0\}$ and $X \in \partial\{u_{\lambda-\delta} = 0\}$, then

$$0 \leq \frac{h(X') - h(X)}{|X' - X|} \leq \frac{1}{\varepsilon_c}.$$

Now as $\delta \to 0$, we get

$$0 \leq \limsup_{\substack{h(X') > h(X) \\ h(X') \to \lambda \\ h(X) \to \lambda}} \frac{h(X') - h(X)}{|X' - X|} \leq |\nabla u_\lambda|_{L^\infty(\partial\Omega)}.$$

where we have used the continuity of the map $(X, \lambda) \mapsto \nabla u_\lambda(X)$. In particular, we conclude that

$$\mathrm{Lip}(h) \leq \sup_{\lambda \in [0,+\infty)} |\nabla u_\lambda|_{L^\infty(\partial\Omega)} = |\nabla u_\infty|_{L^\infty(\partial\Omega)} < +\infty,$$

where the function u_∞ satisfies

$$\begin{cases} \Delta u_\infty = 1 & \text{on } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

By definition of v^ε, if $X_1 \in \partial\Omega_{(-\varepsilon\delta)}$, there exists $X_2 \in \partial\Omega$ such that

$$|X_2 - X_1| = \varepsilon\delta \quad \text{and} \quad [X_1, X_2] \subset \overline{\Omega} \setminus \Omega_{(-\varepsilon\delta)}.$$

Moreover,

$$\begin{aligned} v^\varepsilon(X_1) &= \sup_{Y \in B_{\varepsilon\delta}(X_1)} u_{\lambda-\delta}(Y) \\ &\leq \lambda - \delta \\ &= u_\lambda(X_2) - \delta \\ &= u_\lambda(X_1) - \delta + (u_\lambda(X_2) - u_\lambda(X_1)) \\ &\leq u_\lambda(X_1) - \delta\left(1 - \varepsilon|\nabla u_\lambda|_{L^\infty(\overline{\Omega}\setminus\Omega_{(-\varepsilon\delta)})}\right). \end{aligned}$$

Consequently,

$$v^\varepsilon < u_\lambda \quad \text{on} \quad \partial\Omega_{(-\varepsilon\delta)} \quad \text{while} \quad \varepsilon < \varepsilon_c = \frac{1}{|\nabla u_\lambda|_{L^\infty(\overline{\Omega}\setminus\Omega_{(-\varepsilon\delta)})}}. \tag{5.3}$$

Here the fact that v^ε is a subsolution for the obstacle problem means the following.

Lemma 5.12. *The function v^ε satisfies*

$$\begin{cases} \Delta v^\varepsilon \geq 1 & \text{on } \{v^\varepsilon > 0\} \cap \Omega_{(-\varepsilon\delta)} \\ v^\varepsilon \geq 0 & \\ \text{if } v^\varepsilon(X_0) = 0, & \text{then } v^\varepsilon(X_0 + X) \leq CX^2 \quad \text{with} \quad C = \frac{1}{2}|\nabla^2 u|_{L^\infty(\Omega)}. \end{cases}$$

Now let us consider the open set where v^ε is bigger than u_λ:

$$\omega := \{v^\varepsilon > u_\lambda\} \cap \Omega_{(-\varepsilon\delta)}.$$

We want to prove that $\omega = \emptyset$ while $\varepsilon < \varepsilon_c$. Let us assume on the contrary that $\omega \neq \emptyset$. Then from Lemma 5.12 and the maximum principle, we deduce that the maximum

$$\max_\omega (v^\varepsilon - u_\lambda) > 0$$

is reached on

$$\partial\omega \cup ((\partial\{u_\lambda = 0\}) \cap \omega).$$

But

$$\partial\omega \subset \left(\partial\Omega_{(-\varepsilon\delta)}\right) \cup \{v^\varepsilon = u_\lambda\}$$

and from (5.3) we get

$$v^\varepsilon - u_\lambda \leq 0 \quad \text{on } \partial\omega \quad \text{while} \quad \varepsilon < \varepsilon_c.$$

We can resume what we have proved in the following.

Lemma 5.13.

$$\begin{cases} v^\varepsilon < u_\lambda \quad on \quad \partial\Omega_{(-\varepsilon\delta)} \\ \max_{\Omega_{(-\varepsilon\delta)}}(v^\varepsilon - u_\lambda) \leq \max_{\partial\{u_\lambda=0\}} v^\varepsilon \end{cases} \qquad \textit{while} \quad \varepsilon < \varepsilon_c.$$

We now apply a continuity method in three steps:
Step 1: Initialization. By the continuity of h (Proposition 5.10) and equality (5.2), there exists an $\varepsilon_1 > 0$ small enough (and in particular smaller than ε_c) such that

$$d(\{u_\lambda = 0\}, \{u_{\lambda-\delta} > 0\}) \geq \varepsilon_1\delta.$$

In particular, we get that

$$v^\varepsilon = 0 \quad \text{on} \quad \partial\{u_\lambda = 0\} \quad \text{for} \quad \varepsilon \leq \varepsilon_1,$$

and from Lemma 5.13, we have

$$v^\varepsilon \leq u_\lambda \quad \text{on} \quad \Omega_{(-\varepsilon\delta)} \quad \text{for} \quad \varepsilon \leq \varepsilon_1.$$

Step 2: Continuation. Let

$$\varepsilon^* = \sup\left\{\varepsilon' \in [0, \varepsilon_c],\ v^\varepsilon \leq u_\lambda \text{ on } \Omega_{(-\varepsilon\delta)} \text{ for all } \varepsilon \leq \varepsilon'\right\}.$$

In particular,

$$v^{\varepsilon^*} \leq u_\lambda \quad \text{on} \quad \Omega_{(-\varepsilon^*\delta)}.$$

If $\varepsilon^* < \varepsilon_c$, then from Lemma 4.13 and the maximum principle, we get

$$v^{\varepsilon^*} < u_\lambda \quad \text{on} \quad \left\{v^{\varepsilon^*} > 0\right\} \cap \Omega_{(-\varepsilon^*\delta)}. \tag{5.4}$$

On the other hand, we have

$$\left\{v^{\varepsilon^*} = 0\right\} \supset \{u_\lambda = 0\}.$$

We will prove that

$$\partial\{u_\lambda = 0\} \cap \partial\left\{v^{\varepsilon^*} > 0\right\} \neq \emptyset. \tag{5.5}$$

If not, we have

$$d\left(\partial\{u_\lambda = 0\}, \left\{v^{\varepsilon^*} > 0\right\}\right) = \eta > 0,$$

and then

$$\begin{aligned} v^{\varepsilon^*+s}(X) &= \sup_{Y \in B_{(\varepsilon^*+s)\delta}(X)} u_{\lambda-\delta}(Y) \\ &= \sup_{Y \in B_{s\delta}(X)} \sup_{Y' \in B_{\varepsilon^*\delta}(Y)} u_{\lambda-\delta}(Y') \\ &= \sup_{Y \in B_{s\delta}(X)} v^{\varepsilon^*}(Y). \end{aligned}$$

Then

$$v^{\varepsilon^*+s} = 0 \quad \text{on} \quad \partial\{u_\lambda = 0\} \quad \text{if} \quad s < \frac{\eta}{\delta} \quad \text{and} \quad \varepsilon^* + s < \varepsilon_c,$$

which by Lemma 5.13 would give a contradiction to the definition of ε^*. Then (5.5) is true and there exists

$$X_3 \in \partial\{u_\lambda = 0\} \cap \partial\left\{v^{\varepsilon^*} > 0\right\}.$$

Moreover, there exists

$$X_4 \in \partial\{u_{\lambda-\delta} = 0\} \quad \text{such that} \quad |X_4 - X_3| = \varepsilon^*\delta.$$

As a consequence,

$$v^{\varepsilon^*} > 0 \quad \text{on} \quad B_{\varepsilon^*\delta}(X_4).$$

Using Lemma 5.13, we can resume the properties of $u_\lambda - v^{\varepsilon^*}$ on the ball $B_{\varepsilon^*\delta}(X_4)$ by

$$\begin{cases} \Delta(u_\lambda - v^{\varepsilon^*}) \leq 0 & \text{on } B_{\varepsilon^*\delta}(X_4) \\ u_\lambda - v^{\varepsilon^*} > 0 & \text{on } B_{\varepsilon^*\delta}(X_4) \\ u_\lambda(X_3) = v^{\varepsilon^*}(X_3) = 0 & \text{with } X_3 \in \partial B_{\varepsilon^*\delta}(X_4). \end{cases}$$

The Hopf lemma implies

$$\frac{d}{dn}(u_\lambda - v^{\varepsilon^*})(X_3) > 0 \quad \text{with} \quad n = \frac{X_3 - X_4}{|X_3 - X_4|}.$$

This is in contradiction with the fact that the nonnegative functions u_λ and v^{ε^*} satisfy

$$\begin{cases} \nabla u_\lambda(X_3) = 0 & \text{because } u_\lambda \in C^{1,1} \\ \nabla v^{\varepsilon^*}(X_3) = 0 & \text{because } v^{\varepsilon^*}(X_3 + X) \leq CX^2. \end{cases}$$

Step 3: Conclusion. As a consequence, we get

$$\varepsilon^* = \varepsilon_c.$$

Proof of Lemma 5.12. This lemma is a straightforward consequence of the following.

Lemma 5.14 ((T. Kato [K72], Variant of Kato's Inequality))**.** *Let two Lipschitz functions u_i for $i = 1, 2$, which satisfy $u_i \geq 0$ on ω, $\Delta u_i \geq 0$ on ω, and*

$$\Delta u_i \geq 1 \quad \text{on } \{u_i > 0\} \cap \omega. \tag{5.6}$$

Then $v = \sup(u_1, u_2)$ satisfies

$$\Delta v \geq 1 \quad \text{on } \{v > 0\} \cap \omega.$$

And Lemma 5.14 can be proved using the original mollification argument of T. Kato in [K72]. □

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